

# Bi-Sobolev homeomorphisms $f$ with $Df$ and $Df^{-1}$ of low rank using laminates

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## Abstract

Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set. Given  $1 \leq m_1, m_2 \leq n - 2$ , we construct a homeomorphism  $f : \Omega \rightarrow \Omega$  that is Hölder continuous,  $f$  is the identity on  $\partial\Omega$ , the derivative  $Df$  has rank  $m_1$  a.e. in  $\Omega$ , the derivative  $Df^{-1}$  of the inverse has rank  $m_2$  a.e. in  $\Omega$ ,  $Df \in W^{1,p}$  and  $Df^{-1} \in W^{1,q}$  for  $p < \min\{m_1 + 1, n - m_2\}$ ,  $q < \min\{m_2 + 1, n - m_1\}$ . The proof is based on convex integration and laminates. We also show that the integrability of the function and the inverse is sharp.

## 1 Introduction

In this paper we prove that there exists a bi-Sobolev homeomorphism  $f : \Omega \rightarrow \mathbb{R}^n$  whose derivative  $Df$  has rank equal to  $m_1$  at a.e. point and  $\text{rank}(Df^{-1}(y)) = m_2$  for a.e.  $y \in f(\Omega)$ , for given  $m_1, m_2 \leq n - 2$ . This map cannot satisfy the Luzin condition (which states that the image by  $f$  of a set of measure zero also has measure zero).

There are several examples in the literature of Sobolev homeomorphisms that do not satisfy the Luzin condition, see [5], [17]. See, in addition, [19] for some sufficient conditions for its validity.

The pioneering construction by Hencl [17] of Sobolev homeomorphisms with zero Jacobian has been further developed in [6, 10] to construct bi-Sobolev homeomorphisms  $f$  with zero minors of  $Df$  and of  $Df^{-1}$  from some order. See [14] for further state of the art.

As far as we know, there are two methods to obtain this class of pathological maps:

- The method showed in [17] and its developments [10, 6]. All those constructions were based on a careful explicit construction and a limit process to obtain a Cantor set where the Jacobian is supported.
- The method that appears in [14], based on staircase laminates. The use of laminates in relation with integrability issues was initiated in [12], where the so-called staircase laminates were introduced. This technique has turned out to be extremely efficient in a number of unrelated results, such as [27], [25], [7], [8], [3] and [4]; see also [20] and [21],

where rank-one convexity is used to obtain badly integrable maps. In [3] a version of the in-approximation of Müller and Šverák [26] valid for unbounded set was developed. In [14], these ideas were extended to construct extremal homeomorphisms: this was an extra feature that did not appear before.

However, in none of those papers there was a consideration on the inverse map. In particular, since the laminates behind the proof of [14] were supported in a set of non-invertible matrices, here we need to proceed differently. Let  $E_m$  be the set of symmetric matrices of rank  $m$ . Then we construct a suitable sequence of open sets  $E_m^j$  of symmetric positive definite matrices converging to  $E_m$ . Then, when we denote the inverse of a laminate  $\nu = \sum_{i=1}^N \lambda_i \delta_{A_i}$  supported in the set of invertible matrices as

$$\nu^{-1} = \frac{1}{\det(\sum_{i=1}^N \lambda_i A_i)} \sum_{i=1}^N \lambda_i \det(A_i) \delta_{A_i^{-1}}$$

(see Definition 8), we construct a sequence of staircase laminates  $\nu_j$  supported in  $E_{m_1}^j$  such that the inverse laminates  $\nu_j^{-1}$  are supported in  $E_{m_2}^j$ . We run convex integration to find sequences  $f_j$  such that the gradient of  $f_j$  is arbitrarily close to  $\nu_j$ , and the gradient of  $f_j^{-1}$  is arbitrarily close to  $\nu_j^{-1}$ . Then, following the ideas of [26] and [3] we are able to show that both  $f_j$  and  $f_j^{-1}$  converge to maps  $f$  and  $f^{-1}$  whose gradients are supported in  $E_{m_1}$  and  $E_{m_2}$ , respectively, and have the highest possible integrability for a bi-Sobolev homeomorphism with rank of the gradient equal to  $m_1$  and rank of the gradient of its inverse equal to  $m_2$ . This method allows us to obtain a pathological bi-Sobolev map, instead of obtaining just a Sobolev map.

Thus, we show that with laminates it is possible to combine the results of [1, 13, 17]. In fact, we also improve the result of Černý [6], where he constructs a bi-Sobolev homeomorphism  $f$  with all the minors of order  $m_1 + 1$  of  $Df$ , and all the minors of order  $m_2 + 1$  of  $Df^{-1}$  equal to zero almost everywhere, belonging to  $W^{1,p}$  with  $1 \leq p < \min\{\frac{n}{n-m_1}, n - m_2\}$  and with the inverse  $f^{-1}$  in  $W^{1,q}$  with  $1 \leq q < \min\{\frac{n}{n-m_2}, n - m_1\}$ . The bi-Sobolev homeomorphism  $f$  that we construct satisfies that  $Df$  has rank equal to  $m_1$  and  $Df^{-1}$  has rank equal to  $m_2$ , and  $f$  is in  $W^{1,p}$  and  $f^{-1}$  is in  $W^{1,q}$  for  $1 \leq p < \min\{m_1 + 1, n - m_2\}$  and  $1 \leq q < \min\{m_2 + 1, n - m_1\}$ . We also recover a slightly weaker version of the result in [14]; to be precise, here we obtain that  $Df \in L^p$  for all  $p < \text{rank}(Df) + 1$ , whereas in [14] it is obtained that  $Df$  is in the weak  $L^{\text{rank}(Df)+1}$  space.

**Theorem 1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set,  $m_1, m_2 \in \mathbb{N}$ ,  $1 \leq m_1, m_2 \leq n - 2$ ,  $\varepsilon > 0$  and  $\alpha \in (0, 1)$ . Then there exists a convex function  $u : \Omega \rightarrow \mathbb{R}$  whose gradient  $f = \nabla u : \Omega \rightarrow \Omega$  is a bi-Sobolev homeomorphism and satisfies:*

- i)  $f = \text{id}$  on  $\partial\Omega$ .
- ii)  $\|f - \text{id}\|_{C^\alpha(\overline{\Omega})} < \varepsilon$  and  $\|f^{-1} - \text{id}\|_{C^\alpha(\overline{\Omega})} < \varepsilon$ .
- iii)  $\text{rank}(Df(x)) = m_1$  a.e.  $x \in \Omega$  and  $\text{rank}(Df^{-1}(y)) = m_2$  for a.e.  $y \in \Omega$ .

iv)  $Df \in \bigcap_{p < m'_1+1} L^p(\Omega, \mathbb{R}^{n \times n})$  and  $Df^{-1} \in \bigcap_{q < m'_2+1} L^q(\Omega, \mathbb{R}^{n \times n})$ ,

where for  $i = 1, 2$  we have

$$m'_i = \begin{cases} m_i & \text{if } m_1 + m_2 \leq n - 1, \\ m_i - (m_1 + m_2 - n + 1) & \text{if } m_1 + m_2 \geq n. \end{cases}$$

Notice that, as explained in [17], using the area formula for Sobolev mappings ([16]) we have that this kind of homeomorphisms sends a set of full measure to a null set, and a null set to a set of full measure.

The key to see that Theorem 1 is sharp is the following result, whose proof is based in the one of [18, Th. 4] and, in fact, generalizes that result.

**Theorem 2.** *Let  $f : \Omega \rightarrow \mathbb{R}^n$  be a continuous bi-Sobolev map such that  $f^{-1} \in W^{1,q}(f(\Omega), \mathbb{R}^n)$  and for a measurable set  $E \subset \Omega$  we have  $Jf = 0$  almost everywhere on  $E$ . Let  $m \in \mathbb{N}$  and assume one of the following:*

a)  $q = m$  and  $f^{-1}$  is Hölder continuous.

b)  $q > m$ .

Then  $\text{rank}(Df) \leq n - m - 1$  almost everywhere on  $E$ .

Observe that this implies that given a continuous bi-Sobolev map  $f \in W^{1,p}$  with  $p > n - 1$ , we have  $Jf^{-1}(y) \neq 0$  a.e.  $y \in f(\Omega)$ .

Using the last theorem and Theorem 12 of [14] we obtain the following two results. We think that the next one has interest by itself: it gives an upper bound for the sum of the integrabilities of  $Df$  and  $Df^{-1}$  for a bi-Sobolev homeomorphism with Jacobian equal to zero almost everywhere.

**Theorem 3.** *Let  $f : \Omega \rightarrow \mathbb{R}^n$  be a bi-Sobolev homeomorphism such that  $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ ,  $f^{-1} \in W^{1,q}(f(\Omega), \mathbb{R}^n)$ ,  $f|_{\partial\Omega} = \text{id}|_{\partial\Omega}$  and  $Jf = 0$  almost everywhere in  $\Omega$ . Then  $p + q \leq n + 1$ . Moreover, if  $p + q = n + 1$  then  $p, q \in \mathbb{N}$  and  $f, f^{-1} \notin C^\alpha$  for any  $\alpha \in (0, 1)$ .*

We do not know whether there exists a bi-Sobolev homeomorphism that attains the equality  $p + q = n + 1$ .

Finally, the next theorem shows that Theorem 1 is sharp and gives bounds for the integrability of  $Df$  and  $Df^{-1}$  depending on the rank of  $Df$  and  $Df^{-1}$ .

**Theorem 4.** *Let  $m_1, m_2 \in \mathbb{N}$  be such that  $m_1, m_2 \leq n - 1$ , let  $f : \Omega \rightarrow \mathbb{R}^n$  be a bi-Sobolev homeomorphism such that  $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ ,  $f^{-1} \in W^{1,q}(f(\Omega), \mathbb{R}^n)$ ,  $f|_{\partial\Omega} = \text{id}|_{\partial\Omega}$ ,  $\text{rank}(Df) \leq m_1$ ,  $\text{rank}(Df^{-1}) \leq m_2$  almost everywhere in  $\Omega$  and suppose that there exist measurable  $A, B \subset \Omega$  with  $|A|, |B| > 0$ ,  $\text{rank}(Df) = m_1$  on  $A$  and  $\text{rank}(Df^{-1}) = m_2$  on  $B$ . Then  $p \leq \min\{m_1 + 1, n - m_2\}$  and  $q \leq \min\{m_2 + 1, n - m_1\}$ .*

Observe that this theorem implies that  $p + q \leq n + 1$ .

The structure of the paper is as follows. Section 2 presents the general notation of the paper. In Section 3 we prove Theorems 2, 3 and 4. Section 4 shows the construction of the laminates in the simplest case of dimension three. This section gives an idea of how we prove Theorem 1 and helps to understand the actual proof. Section 5, which is the bulk of the paper, constructs a sequence of laminates that converges to the probability measure sketched in Section 3, but in general dimensions, as well as a sequence of functions whose gradients approximate the laminates.

## 2 General notation

We explain the general notation used throughout the paper, most of which is standard.

In the whole paper,  $\Omega$  is an open, non-empty bounded set of  $\mathbb{R}^n$ .

We denote by  $\mathbb{R}^{n \times n}$  the set of  $n \times n$  matrices, by  $\Gamma_+$  its subset of symmetric positive semidefinite matrices, by  $SO(n) \subset \mathbb{R}^{n \times n}$  the orthogonal matrices with determinant 1, and by  $I$  the identity matrix.

Given  $A_i \in \mathbb{R}^{n \times n}$ , the measure  $\delta_{A_i}$  is the Dirac delta at  $A_i$ . The barycenter of the probability measure  $\nu = \sum_{i=1}^N \alpha_i \delta_{A_i}$  is  $\bar{\nu} = \sum_{i=1}^N \alpha_i A_i$ .

Given  $A \in \mathbb{R}^{n \times n}$ , let  $\sigma_1(A) \leq \dots \leq \sigma_n(A)$  denote its singular values. If the matrix  $A$  is clear from the context, we will just indicate their singular values as  $\sigma_1, \dots, \sigma_n$ . In fact, in this paper we will always deal with  $A \in \Gamma_+$ , so its eigenvalues coincide with its singular values. Its components are written  $A_{\alpha, \beta}$  for  $\alpha, \beta \in \{1, \dots, n\}$ . Its operator norm is denoted by  $|A|$ , which coincides with  $\sigma_n(A)$ . The norm of a  $v \in \mathbb{R}^n$  is also denoted by  $|v|$ .

Given  $a_1, \dots, a_n \in \mathbb{R}$  the matrix  $\text{diag}(a_1, \dots, a_n) \in \mathbb{R}^{n \times n}$  is the diagonal matrix with diagonal entries  $a_1, \dots, a_n$ .

We will use the symbol  $\lesssim$  when there exists a constant depending only on  $n, m_1$  and  $m_2$  such that the left hand side is less than or equal to the constant times the right hand side. Sometimes, the left hand side could be negative.

Given a set  $E \subset \mathbb{R}^n$ , we denote its characteristic function by  $\chi_E$ . We write  $\#E$  for the number of elements of  $E$ . When it is measurable, its Lebesgue measure is denoted by  $|E|$  and we use  $\mathcal{H}^m(E)$  for its Hausdorff measure of dimension  $m$ .

Given  $a \in \mathbb{R}$ , its integer part is denoted by  $\lfloor a \rfloor$  and we denote by  $\lceil a \rceil$  its ceiling function.

Given  $E \subset \mathbb{R}^n$ ,  $\alpha \in (0, 1]$  and a function  $f : E \rightarrow \mathbb{R}^n$ , we denote the Hölder seminorm, supremum norm and Hölder norm, respectively, as

$$|f|_{C^\alpha(E)} := \sup_{\substack{x_1, x_2 \in E \\ x_1 \neq x_2}} \frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|^\alpha}, \quad \|f\|_{L^\infty(E)} = \sup_{x \in E} |f(x)|,$$

$$\|f\|_{C^\alpha(E)} := |f|_{C^\alpha(E)} + \|f\|_{L^\infty(E)}.$$

We will write  $f \in C^\alpha(E, \mathbb{R}^n)$  when  $\|f\|_{C^\alpha(E)} < \infty$ . Note that, if  $f$  is continuous up to the boundary, the above norms and seminorms in  $E$  coincide with those in  $\overline{E}$ . In particular, we

will identify  $C^\alpha(E, \mathbb{R}^n)$  with  $C^\alpha(\overline{E}, \mathbb{R}^n)$ , the set of Hölder functions of exponent  $\alpha$ . Of course, if  $\alpha = 1$ , they are Lipschitz.

The identity function is denoted by  $\text{id}$  and the Sobolev space from  $\Omega$  to  $\mathbb{R}^n$  is denoted, alternatively, by  $W^{1,p}$ ,  $W^{1,p}(\Omega)$  or  $W^{1,p}(\Omega, \mathbb{R}^n)$ .

Given  $f : A \rightarrow \mathbb{R}^n$ , where  $A$  is a subset of an  $m$ -dimensional affine space of  $\mathbb{R}^n$ , we say that it satisfies the  $m$ -dimensional Luzin condition, also known as condition  $N$ , if for every  $E \subset A$  with  $\mathcal{H}^m(E) = 0$ , then  $\mathcal{H}^m(f(E)) = 0$ .

For  $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ , we denote the Jacobian of  $f$  by  $Jf$  and, for  $k \in \{1, \dots, n\}$ , the  $k$ -dimensional Jacobian of  $f$  by  $J_k f$ , i.e.,  $J_k f$  is equal to  $(\sum_M |M|^2)^{\frac{1}{2}}$ , where the sum runs over all the minors of  $Df$  of order  $k$ .

We will say that a continuous mapping  $f : \overline{\Omega} \rightarrow \mathbb{R}^n$  is *piecewise affine* if there exists a countable family  $\{\Omega_i\}_{i \in \mathbb{N}}$  of pairwise disjoint open subsets of  $\Omega$  such that  $f|_{\Omega_i}$  is affine for all  $i \in \mathbb{N}$ , and

$$\left| \Omega \setminus \bigcup_{i \in \mathbb{N}} \Omega_i \right| = 0.$$

Note that  $\{\Omega_i\}_{i \in \mathbb{N}}$  need not be locally finite. Given  $S \subset \mathbb{R}^{n \times n}$  a set of invertible matrices we denote by  $S^{-1}$  the set

$$\{A^{-1} : A \in S\}.$$

### 3 Limitations to the integrability of $f$ and $f^{-1}$

In this section we prove Theorems 2, 3 and 4. The proof of Theorem 2 follows that in [18, Th. 4].

*Proof of Theorem 2.* Suppose that  $\text{rank}(Df) \geq n - m$  in a set  $A \subset E$  of positive measure. Then  $|J_{n-m} f| > 0$  on  $A$ , and without loss of generality we can assume that  $Jf = 0$  on  $A$  and that  $f$  is Lipschitz on  $A$ , see [11, Section 6.6, Th.3].

For each  $I \subset \{1, \dots, n\}$  with  $|I| = n - m$  let  $\pi_I : \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$  be the projection that sends  $x = (x_1, \dots, x_n)$  to  $(x_{i_1}, \dots, x_{i_{n-m}})$ , where  $I = \{i_1, \dots, i_{n-m}\}$  and  $i_1 < \dots < i_{n-m}$ . Define  $h_I = \pi_I \circ f$  and set  $P(z) = \pi_I^{-1}(z) \cap f(\Omega)$  for  $z \in \mathbb{R}^{n-m}$ . Since  $|J_{n-m} f| > 0$  in  $A$ , there exists  $I \subset \{1, \dots, n\}$  such that  $|I| = n - m$  and  $|J_{n-m} h_I| > 0$  on a subset of  $A$  of positive measure, still called  $A$ . Since  $f$  is Lipschitz on  $A$ , we can use the area formula (see, e.g. [11], [15], [2]) to conclude that

$$|f(A)| = 0.$$

Using the coarea formula we get

$$0 < \int_A |J_{n-m} h_I| = \int_{\mathbb{R}^{n-m}} \mathcal{H}^m(\{x \in A : h_I(x) = z\}) dz.$$

Hence, for a set  $F \subset \mathbb{R}^{n-m}$  of positive measure we have that  $\forall z \in F$ ,

$$(1) \quad \mathcal{H}^m(f^{-1}(f(A) \cap P(z))) = \mathcal{H}^m(\{x \in A : h_I(x) = z\}) > 0.$$

In the equality we have used that  $f$  is injective. On the other hand, since  $|f(A)| = 0$  it follows that for  $\mathcal{H}^{n-m}$ -almost every  $z \in \mathbb{R}^{n-m}$  we get  $\mathcal{H}^m(f(A) \cap P(z)) = 0$ .

As  $f^{-1} \in W^{1,q}$ , we have that  $f^{-1} \in W^{1,q}(P(z))$  for  $\mathcal{H}^{n-m}$ -almost every  $z \in \mathbb{R}^{n-m}$ , so under either option  $a)$  or  $b)$ ,  $f^{-1}|_{P(z)}$  satisfies the  $m$ -dimensional Luzin (N) condition. The proof under  $a)$  is due to [22, Theorem 1.1] (with  $\lambda = 0$  in the notation there), while the proof under  $b)$  is classical [24]. Therefore, for  $\mathcal{H}^{n-m}$ -almost every  $z \in \mathbb{R}^{n-m}$ , we obtain

$$\mathcal{H}^m(f^{-1}(f(A) \cap P(z))) = 0,$$

and we have a contradiction with (1). □

Now using Theorem 2 we are able to prove Theorem 3.

*Proof of Theorem 3.* Define  $m_p = \lceil p \rceil - 1$  and  $m_q = \lceil q \rceil - 1$ . Observe that  $m_p < p \leq m_p + 1$  and  $m_q < q \leq m_q + 1$ . Then, using Theorem 2, we have that  $\text{rank } Df \leq n - m_q - 1$  almost everywhere on  $\Omega$ . On the other hand, using [14, Th. 12], we get that  $\text{rank } Df \geq m_p$  in a subset of  $\Omega$  of positive measure; we also obtain  $\text{rank } Df^{-1} \geq m_q$ . Therefore, we have

$$m_p \leq n - m_q - 1.$$

Hence

$$(2) \quad p + q \leq m_p + m_q + 2 \leq n + 1.$$

If  $p + q = n + 1$  then  $p = m_p + 1$  and  $q = m_q + 1$ . If, in addition,  $f^{-1}$  or  $f$  were Hölder continuous then, by Theorem 2,  $\text{rank}(Df) \leq n - q - 1$  or  $\text{rank}(Df^{-1}) \leq n - p - 1$ , but this contradicts  $\text{rank}(Df) \geq m_p = p - 1$  in the first case and  $\text{rank}(Df^{-1}) \geq m_q = q - 1$  in the second case. □

*Proof of Theorem 4.* First, we observe that thanks to [14, Th. 12] we obtain  $p \leq m_1 + 1$  and  $q \leq m_2 + 1$ .

On the other hand, denote  $m = \lceil q \rceil - 1$ , then  $m < q \leq m + 1$ , and using Theorem 2 with  $E = A$  we obtain  $\text{rank}(Df) \leq n - m - 1 \leq n - q$  almost everywhere on  $A$ , so  $q \leq n - m_1$ . In the same way, we get  $p \leq n - m_2$ . The theorem follows. □

## 4 Construction of the laminate in dimension three

The next definition introduces the concept of laminate of finite order [9, 27, 26, 3].

**Definition 5.** *The family  $\mathcal{L}(\mathbb{R}^{n \times n})$  of laminates of finite order is the smallest family of probability measures in  $\mathbb{R}^{n \times n}$  with the properties:*

- i)  $\mathcal{L}(\mathbb{R}^{n \times n})$  contains all the Dirac masses.*

ii) If  $\sum_{i=1}^N \lambda_i \delta_{A_i} \in \mathcal{L}(\mathbb{R}^{n \times n})$  and  $A_N = \lambda B + (1 - \lambda)C$ , where  $\lambda \in [0, 1]$  and  $\text{rank}(B - C) = 1$ , then the probability measure

$$\sum_{i=1}^{N-1} \lambda_i \delta_{A_i} + \lambda_N (\lambda \delta_B + (1 - \lambda) \delta_C)$$

is also in  $\mathcal{L}(\mathbb{R}^{n \times n})$ .

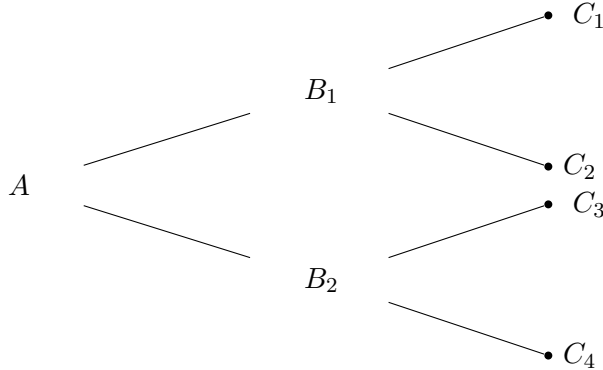
Since in this work we will only use laminates of finite order, for simplicity they will be just called *laminates*. Note that they are a convex combination of Dirac masses.

The next lemma gives us a characterization of the laminates of finite order.

**Lemma 6.** *For every laminate of finite order  $\nu = \sum_{i=1}^N \lambda_i \delta_{A_i}$  there exists a family  $\{\nu_j\}_{j=1}^N$  of laminates of finite order, such that*

- $\nu_1 = \delta_{\overline{\nu}}$ ,
- for  $j \in \{1, \dots, N - 1\}$ ,  $\nu_{j+1}$  is obtained from  $\nu_j$  using once ii) of Definition 5,
- $\nu_N = \nu$ .

Lemma 6 has an elementary proof. It is illustrated in the following example. We have the splits



and  $\nu = \sum_{i=1}^4 \lambda_i \delta_{C_i}$ . Then the laminates  $\{\nu_i\}_{i=1}^4$  can be chosen to be the following:

$$\nu_4 = \nu,$$

$$\nu_3 = (\lambda_1 + \lambda_2) \delta_{B_1} + \lambda_3 \delta_{C_3} + \lambda_4 \delta_{C_4},$$

$$\nu_2 = (\lambda_1 + \lambda_2) \delta_{B_1} + (\lambda_3 + \lambda_4) \delta_{B_2},$$

$$\nu_1 = \delta_A.$$

From Lemma 6 we obtain the following corollary that will be used throughout the paper to prove that some measures are laminates.

**Corollary 7.** Let  $\nu = \sum_{i=1}^N \lambda_i \delta_{A_i}$  and  $\{\nu_{A_i}\}_{i=1}^N$  be laminates of finite order such that

$$\overline{\nu}_{A_i} = A_i \text{ with } A_i \text{ being all different.}$$

Then, the probability measure

$$\nu' = \sum_{i=1}^N \lambda_i \nu_{A_i} = \nu + \sum_{i=1}^N \nu(A_i) [\nu_{A_i} - \delta_{A_i}]$$

is also a laminate of finite order.

**Definition 8.** We define the inverse laminate of a laminate  $\nu = \sum_{i=1}^N \lambda_i \delta_{A_i}$  supported in the set of positive definite matrices as

$$\nu^{-1} = \frac{1}{\det(\overline{\nu})} \sum_{i=1}^N \lambda_i \det(A_i) \delta_{A_i^{-1}}.$$

This definition, which seems to be new, arises from the fact that if  $\sum_{i=1}^N \lambda_i \delta_{A_i}$  is a laminate supported in the set of positive definite matrices and  $f$  is a piecewise affine Sobolev homeomorphism satisfying

$$|E_i| = \lambda_i \text{ and } f|_{\partial E_i} = A_i \quad \text{for } i \in \{1, \dots, N\},$$

for

$$E_i = \{x \in \Omega : |Df(x) - A_i| < \delta\},$$

and some  $\delta > 0$ , then we get

$$|f(E_i)| = \int_{E_i} \det Df(x) dx = \int_{E_i} \det A_i dx = \lambda_i \det A_i \quad \text{for } i \in \{1, \dots, N\},$$

and hence, there exists  $\delta' > 0$  such that  $f^{-1}$  satisfies

$$|\{y \in f(\Omega) : |Df^{-1}(y) - A_i^{-1}| < \delta'\}| = |f(E_i)| = \lambda_i \det A_i.$$

Although we will not use it,  $\nu^{-1}$  is also a laminate; this can be shown using the fact that  $\det$  is rank-one linear.

In this section we construct the sequence of laminates  $\nu_j$  of finite order that is behind the proof of Theorem 1 in the case  $n = 3$  and  $m_1 = m_2 = 1$ . The actual proof will consist in approximating  $\nu_k$  with laminates of finite order supported in the set of positive definite matrices, then use Proposition 18 to obtain homeomorphisms that are close to the approximate laminates and such that their inverse are close to the inverse of that laminates, then paste the obtained homeomorphisms to construct a homeomorphism in the whole domain and, finally, a limit passage will yield the homeomorphism  $f$  of Theorem 1.

Although this section is not strictly necessary for the proof of Theorem 1, it will help the reader to follow the construction of Section 5.



To define  $\nu_j$  we need to define the following sets. For  $i, k \in \mathbb{N}$  let

$$(3) \quad A_k^i = \{A \in \Gamma_+ : \sigma_j(A) = k^{-1} \text{ for } j \in \{1, 2\} \text{ and } \sigma_3(A) \in \{i-1, i\} \setminus \{0\}\},$$

$$(4) \quad B_k^i = \{A \in \Gamma_+ : \sigma_1(A) \in \{k^{-1}, (k-1)^{-1}\} \setminus \{\infty\} \text{ and } \sigma_j(A) = i \text{ for } j \in \{2, 3\}\} \setminus \{I\}.$$

Observe that a matrix  $A \in \Gamma_+ \setminus \{I\}$  belongs to  $A_k^i$  if and only if  $A^{-1}$  is in  $B_i^k$ .

The laminates  $\nu_j$  that we will construct will satisfy the following:

$$(a) \quad \bar{\nu}_j = I,$$

$$(b) \quad \text{supp}(\nu_j) \subset \bigcup_{i=1}^j A_j^i \cup B_i^j.$$

(c) For all  $\varepsilon > 0$  there exists a bounded family of constants  $\{C_j\}_{j \in \mathbb{N}}$ , such that, for all  $j \in \mathbb{N}$ ,

$$\nu_j(A_j^i) \leq C_j i^{-3+\varepsilon} \text{ and } \nu_j(B_i^j) \leq C_j i^{-2+\varepsilon} j^{-2}.$$

When we approximate these laminates by functions,  $f_j$ , and then pass to the limit, we obtain a bi-Sobolev homeomorphism  $f$  that is the identity on the border due to (a); by (b) and (c) we get for  $f_j$

$$Df_j \in \bigcup_{i=1}^j (A_j^i \cup B_i^j) + B(0, r_j) \text{ with } r_j \rightarrow 0,$$

$$|\{x : Df_j(x) \in \bigcup_{i=1}^j B_i^j + B(0, r_j)\}| \rightarrow 0,$$

and, for the inverse, using that  $A \in A_j^i$  if and only if  $A^{-1} \in B_i^j$ , we obtain

$$Df_j^{-1} \in \bigcup_{i=1}^j (A_j^i \cup B_i^j) + B(0, r'_j) \text{ with } r'_j \rightarrow 0,$$

$$|\{y : Df_j^{-1}(y) \in \bigcup_{i=1}^j B_i^j + B(0, r'_j)\}| \rightarrow 0.$$

So, the ranks of  $Df$  and  $Df^{-1}$  are equal to 1 almost everywhere. Moreover, with (c) we obtain the following. Let  $t, \varepsilon > 0$ , and pick  $j \in \mathbb{N}$ ,  $j > t$  and big enough; then

$$\nu_j(\{A \in \mathbb{R}^{3 \times 3} : |A| > t\}) \lesssim \sum_{i=[t]}^j i^{-3+\varepsilon} + j^{-2} \sum_{i=1}^j i^{-2+\varepsilon} \lesssim t^{-2+\varepsilon},$$

$$\begin{aligned} \nu_j^{-1}(\{A \in \mathbb{R}^{3 \times 3} : |A| > t\}) &\lesssim \sum_{i=1}^j i^{-3+\varepsilon} \sup_{M \in A_j^i} \{\det M\} + j^{-2} \sum_{i=[t]}^j i^{-2+\varepsilon} \sup_{M \in B_i^j} \{\det M\} \\ &\leq j^{-2} \sum_{i=1}^j i^{-2+\varepsilon} + 2 \sum_{i=[t]}^j i^{-3+\varepsilon} \lesssim t^{-2+\varepsilon}. \end{aligned}$$

Therefore, this gives us that  $Df, Df^{-1} \in \bigcap_{p < 2} W^{1,p}$ , respectively.

The laminates  $\nu_j$  are defined inductively as follows. We begin with  $\nu_1 = I$ . It is clear that  $\nu_1$  satisfies (a), (b) and (c). Now, let

$$(5) \quad \nu_j = \sum_{k=1}^N \lambda_k \delta_{A_k} \in \mathcal{L}(\mathbb{R}^{n \times n}),$$

with  $A_k \in \bigcup_{i=1}^j (A_j^i \cup B_i^j)$ , all different. For each  $A \in \text{supp}(\nu_j)$  we are going to construct a laminate  $\nu_A$ , whose support is in  $\bigcup_{i=1}^{j+1} (A_{j+1}^i \cup B_i^{j+1})$ . To do that, we need the following two lemmas.

**Lemma 9.** *Let  $A \in A_k^i$ . Then there exists a laminate of finite order  $\nu$  such that*

- $\bar{\nu} = A$ ,
- $\text{supp}(\nu) \subset A_{k+1}^i \cup B_{k+1}^i$ ,
- $\nu(B_{k+1}^i) \lesssim (k^2 i)^{-1}$ .

**Lemma 10.** *Let  $A \in B_k^i$ . Then there exists a laminate of finite order  $\nu$  such that*

- $\bar{\nu} = A$ ,
- $\text{supp}(\nu) \subset A_k^{i+1} \cup B_k^{i+1}$ ,
- $\nu(A_k^{i+1}) \lesssim i^{-1}$ ,
- $\nu(B_k^{i+1}) - \left(\frac{i}{i+1}\right)^2 \lesssim (ik)^{-2}$ .

We will only prove Lemma 9, the proof of Lemma 10 being analogous.

*Proof of Lemma 9.* Given  $A \in A_k^i$ , without loss of generality, we can assume that  $A = \text{diag}(k^{-1}, k^{-1}, \sigma)$ , with  $\sigma \in \{\max\{i-1, 1\}, i\}$ .

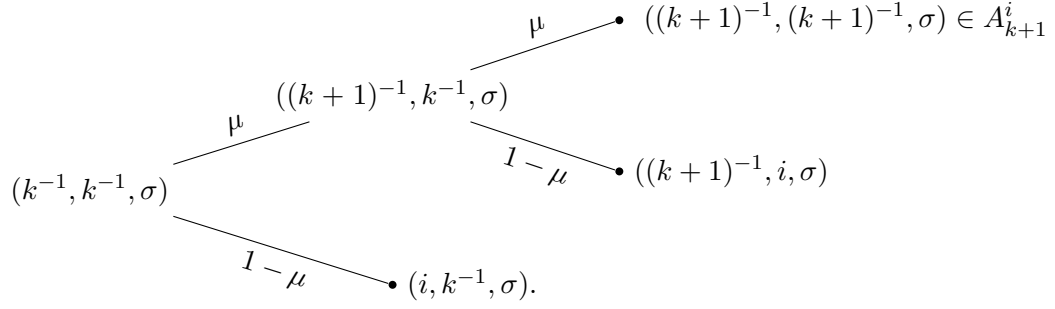
Now, we denote  $\mu = \frac{i-k^{-1}}{i-(k+1)^{-1}}$ ; observe that  $\mu$  satisfies

$$(6) \quad 0 < \mu \leq 1 \quad \text{and} \quad 1 - \mu \leq (k^2 i)^{-1}.$$

Using that

$$(7) \quad k^{-1} = \mu(k+1)^{-1} + (1-\mu)i,$$

we split  $A$  in the following way:



Therefore

$$A = \mu^2 \text{diag}((k+1)^{-1}, (k+1)^{-1}, \sigma) + \mu(1-\mu) \text{diag}((k+1)^{-1}, i, \sigma) \\ + (1-\mu) \text{diag}(i, k^{-1}, \sigma).$$

If  $\sigma = i$  we define

$$\nu = \mu^2 \delta_{\text{diag}((k+1)^{-1}, (k+1)^{-1}, i)} + \mu(1-\mu) \delta_{\text{diag}((k+1)^{-1}, i, i)} + (1-\mu) \delta_{\text{diag}(i, k^{-1}, i)},$$

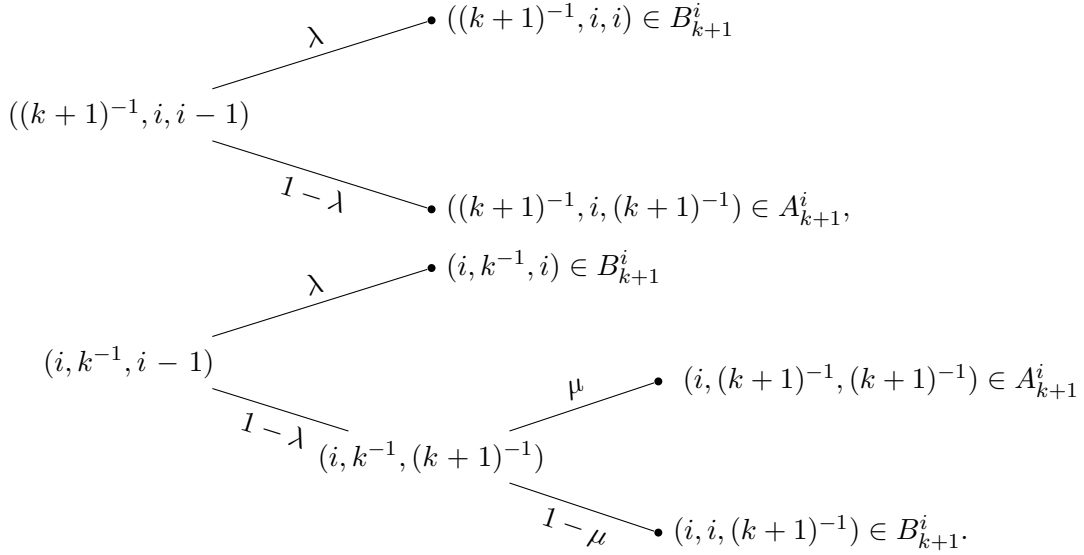
which, clearly, is a laminate supported in  $A_{k+1}^i \cup B_{k+1}^i$ , and by (6), we have that

$$\nu(B_{k+1}^i) \lesssim (k^2 i)^{-1}.$$

If, on the contrary,  $\sigma = i-1$ , we define  $\lambda = \frac{i-1-(k+1)^{-1}}{i-(k+1)^{-1}}$  and using (7) and that

$$i-1 = \lambda i + (1-\lambda)(k+1)^{-1},$$

we do the following splits:



Hence

$$\begin{aligned}
A = & \mu^2 \text{diag}((k+1)^{-1}, (k+1)^{-1}, i-1) + \mu(1-\mu)\lambda \text{diag}((k+1)^{-1}, i, i) \\
& + \mu(1-\mu)(1-\lambda) \text{diag}((k+1)^{-1}, i, (k+1)^{-1}) \\
& + (1-\mu)\lambda \text{diag}(i, k^{-1}, i) + (1-\mu)(1-\lambda)\mu \text{diag}(i, (k+1)^{-1}, (k+1)^{-1}) \\
& + (1-\mu)^2(1-\lambda) \text{diag}(i, i, (k+1)^{-1}),
\end{aligned}$$

and we define

$$\begin{aligned}
\nu = & \mu^2 \delta_{\text{diag}((k+1)^{-1}, (k+1)^{-1}, i-1)} + \mu(1-\mu)\lambda \delta_{\text{diag}((k+1)^{-1}, i, i)} \\
& + \mu(1-\mu)(1-\lambda) \delta_{\text{diag}((k+1)^{-1}, i, (k+1)^{-1})} \\
& + (1-\mu)\lambda \delta_{\text{diag}(i, k^{-1}, i)} + (1-\mu)(1-\lambda)\mu \delta_{\text{diag}(i, (k+1)^{-1}, (k+1)^{-1})} \\
& + (1-\mu)^2(1-\lambda) \delta_{\text{diag}(i, i, (k+1)^{-1})},
\end{aligned}$$

which is a laminate supported in  $A_{k+1}^i \cup B_{k+1}^i$ . Using (6) we obtain

$$\nu(B_{k+1}^i) \leq 1 - \mu^2 \lesssim 1 - \mu \leq (k^2 i)^{-1},$$

and the proof is complete.  $\square$

Now, we can prove the next two lemmas that will give us the laminate  $\nu_A$ .

**Lemma 11.** *Let  $i \in \mathbb{N}$ ,  $i \leq j$  and  $A \in A_j^i$ . Then, there exists a laminate  $\nu$  such that*

- $\bar{\nu} = A$ ,
- $\text{supp}(\nu) \subset \bigcup_{b=0}^{j-i+1} A_{j+1}^{i+b} \cup B_{j+1}^{j+1}$ ,
- $\nu(A_{j+1}^{i+b}) \lesssim j^{-2} \frac{i}{(i+b)^3}$ , for  $b \in \{1, \dots, j-i+1\}$ ,
- $\nu(B_{j+1}^{j+1}) \lesssim j^{-2} i(j+1)^{-2}$ .

**Lemma 12.** *Let  $i \in \mathbb{N}$ ,  $i \leq j$  and  $A \in B_i^j$ . Then, there exists a laminate  $\nu$  such that*

- $\bar{\nu} = A$ ,
- $\text{supp}(\nu) \subset A_{j+1}^{j+1} \cup \bigcup_{b=0}^{j-i+1} B_{i+b}^{j+1}$ ,
- $\nu(A_{j+1}^{j+1}) \lesssim j^{-1}$ ,
- $\nu(B_i^{j+1}) - \left(\frac{j}{j+1}\right)^2 \lesssim j^{-2} i^{-2}$ ,
- $\nu(B_{i+b}^{j+1}) \lesssim ((i+b-1)(j+1))^{-2}$ , for  $b \in \{1, \dots, j-i+1\}$ .

As before, we will only prove Lemma 11 since the proof of Lemma 12 can be obtained in the same form.

*Proof of Lemma 11.* It is enough to construct a family of laminates  $\{\nu'_\ell\}_{\ell=1}^{j-i+2}$  such that

- i)  $\overline{\nu}'_\ell = A$ ,
- ii)  $\text{supp}(\nu'_\ell) \subset \bigcup_{b=0}^{\ell-1} A_{j+1}^{i+b} \cup B_{j+1}^{i+\ell-1}$ ,
- iii)  $\nu'_\ell(A_{j+1}^{i+b}) \lesssim j^{-2} \frac{i}{(i+b)^3}$ , for  $b \in \{1, \dots, \ell-1\}$ ,
- iv)  $\nu'_\ell(B_{j+1}^{i+\ell-1}) \lesssim j^{-2} i(i+\ell-1)^{-2} \left(1 + j^{-2} 8C \sum_{k=1}^{\ell-1} (i+k-1)^{-2}\right)$ ,

and define  $\nu = \nu'_{j-i+2}$ . The constant  $C$  is bigger than those that appear in Lemma 10. Let  $\nu'_1$  be the laminate of Lemma 9; then,  $\nu'_1$  satisfies all the conditions. Now, for  $1 \leq \ell \leq j-i+1$ , given  $\nu'_\ell = \sum_{a=1}^{N_\ell} \lambda_a \delta_{B_a}$  with all the  $B_a$  different, define  $\nu_{B_a}$  as the laminate of Lemma 10 if  $B_a \in B_{j+1}^{i+\ell-1}$  and as  $\delta_{B_a}$  otherwise. Set

$$\nu'_{\ell+1} = \sum_{a=1}^{N_\ell} \lambda_a \nu_{B_a}.$$

It is immediate that  $\nu'_{\ell+1}$  satisfies i), ii), and iii) for  $b \in \{1, \dots, \ell-1\}$ . Thanks to Corollary 7, it is a laminate. Hence, we only have to bound  $\nu'_{\ell+1}(A_{j+1}^{i+\ell})$  and  $\nu'_{\ell+1}(B_{j+1}^{i+\ell})$ . We have

$$\nu'_{\ell+1}(A_{j+1}^{i+\ell}) \lesssim \nu'_\ell(B_{j+1}^{i+\ell-1})(i+\ell-1)^{-1} \lesssim j^{-2} i(i+\ell-1)^{-3} \lesssim j^{-2} \frac{i}{(i+\ell)^3}.$$

For  $j$  big enough,  $j^{-2} 8C \sum_{k=1}^{\infty} (i+k-1)^{-2} \leq 1$ , so

$$\begin{aligned} & 1 + 4C((i+\ell-1)(j+1))^{-2} + (1 + 4C((i+\ell-1)(j+1))^{-2}) j^{-2} 8C \sum_{k=1}^{\ell-1} (i+k-1)^{-2} \\ & \leq 1 + j^{-2} 8C \sum_{k=1}^{\ell} (i+k-1)^{-2}. \end{aligned}$$

Hence,

$$\begin{aligned} \nu'_{\ell+1}(B_{j+1}^{i+\ell}) & \leq \nu'_\ell(B_{j+1}^{i+\ell-1}) \left( \left( \frac{i+\ell-1}{i+\ell} \right)^2 + C((i+\ell-1)(j+1))^{-2} \right) \\ & \lesssim j^{-2} i(i+\ell)^{-2} \left( 1 + 4C((i+\ell-1)(j+1))^{-2} \right. \\ & \quad \left. + (1 + 4C((i+\ell-1)(j+1))^{-2}) j^{-2} 8C \sum_{k=1}^{\ell-1} (i+k-1)^{-2} \right) \\ & \leq j^{-2} i(i+\ell)^{-2} \left( 1 + j^{-2} 8C \sum_{k=1}^{\ell} (i+k-1)^{-2} \right). \end{aligned}$$

□

From  $\nu_j$  as in (5), we construct  $\nu_{j+1}$  as follows. For each  $A \in \text{supp}(\nu_j)$  we define  $\nu_A$  as the laminate of Lemma 11 if  $A \in \bigcup_{i=1}^j A_j^i$ , and as the laminate of Lemma 12 if  $A \in \bigcup_{i=1}^j B_i^j$ . We define  $\nu_{j+1}$  as

$$\nu_{j+1} = \sum_{k=1}^N \lambda_k \nu_{A_k}.$$

It is clear that  $\nu_{j+1}$  satisfies (a) and (b). Pick  $\varepsilon > 0$  small enough to satisfy  $\sum_{a=1}^{\infty} a^{-2+\varepsilon} \leq 2$ . We will prove by induction on  $j$  that  $\nu_{j+1}$  satisfies (c). Let  $C_0 > 0$  be a constant bigger than the constants appearing in Lemmas 11 and 12. Set  $j_\varepsilon \in \mathbb{N}$  such that  $j_\varepsilon^\varepsilon > 20C_0$ . Define

$$C_j = \max_{i \in \{1, \dots, j\}} \left\{ \nu_j(A_j^i) i^{3-\varepsilon}, \nu_j(B_i^j) i^{2-\varepsilon} j^2 \right\} \quad \text{for } j \leq j_\varepsilon,$$

and

$$C_{j+1} = C_j(1 + 12C_0 j^{-2}) \quad \text{for } j \geq j_\varepsilon.$$

It is clear that  $\sup_{j \in \mathbb{N}} C_j < \infty$ , and we have (c) for  $j \leq j_\varepsilon$ .

Then for  $j \geq j_\varepsilon$  and for each  $i \in \{1, \dots, j\}$  we know that the matrices in  $A_{j+1}^i$  can only come from  $\bigcup_{a=1}^i A_j^a$ , and the matrices in  $B_i^{j+1}$  can only come from  $\bigcup_{a=1}^i B_a^j$ ; therefore we obtain

$$\begin{aligned} \nu_{j+1}(A_{j+1}^i) &\leq \nu_j(A_j^i) + \sum_{a=1}^{i-1} \nu_j(A_j^a) C_0 j^{-2} \frac{a}{i^3} \leq C_j \left( i^{-3+\varepsilon} + \sum_{a=1}^{i-1} C_0 j^{-2} a^{-2+\varepsilon} i^{-3} \right) \\ &\leq C_j(1 + 2C_0 j^{-2}) i^{-3+\varepsilon} \leq C_{j+1} i^{-3+\varepsilon}, \end{aligned}$$

$$\begin{aligned} \nu_{j+1}(B_i^{j+1}) &\leq \nu_j(B_i^j) \left( \left( \frac{j}{j+1} \right)^2 + C_0 j^{-2} i^{-2} \right) + \sum_{a=1}^{i-1} \nu_j(B_a^j) C_0 ((i-1)(j+1))^{-2} \\ &\leq C_j \left( i^{-2+\varepsilon} (j+1)^{-2} + C_0 i^{-4+\varepsilon} j^{-4} + 4C_0 i^{-2} j^{-2} (j+1)^{-2} \sum_{a=1}^{i-1} a^{-2+\varepsilon} \right) \\ &\leq C_j i^{-2+\varepsilon} (j+1)^{-2} (1 + 12C_0 j^{-2}) = C_{j+1} i^{-2+\varepsilon} (j+1)^{-2}. \end{aligned}$$

For  $i = j+1$ , since the matrices in  $A_{j+1}^{j+1} \cup B_{j+1}^{j+1}$  can come from any matrix in the support of  $\nu_j$  we get

$$\begin{aligned} \nu_{j+1}(A_{j+1}^{j+1}) &\leq \sum_{a=1}^j \left[ \nu_j(A_j^a) C_0 j^{-2} \frac{a}{(j+1)^3} + \nu_j(B_a^j) C_0 j^{-1} \right] \\ &\leq C_j C_0 \sum_{a=1}^{i-1} [j^{-2} a^{-2+\varepsilon} (j+1)^{-3} + a^{-2+\varepsilon} j^{-3}] \\ &\leq C_j 20C_0 (j+1)^{-3} \leq C_{j+1} (j+1)^{-3+\varepsilon}, \end{aligned}$$

$$\begin{aligned}
\nu_{j+1}(B_{j+1}^{j+1}) &\leq \sum_{a=1}^j [\nu_j(A_j^a)C_0j^{-2}a(j+1)^{-2} + \nu_j(B_a^j)C_0j^{-2}(j+1)^{-2}] \\
&\leq C_j \left( C_0 \sum_{a=1}^{i-1} [j^{-2}a^{-2+\varepsilon}(j+1)^{-2} + a^{-2+\varepsilon}j^{-4}(j+1)^{-2}] \right) \\
&\leq C_j 20C_0(j+1)^{-4} \leq C_{j+1}(j+1)^{-4+\varepsilon},
\end{aligned}$$

and the proof of (a)–(c) is completed.

## 5 Proof of Theorem 1

The sets that we define next are the key of the proof, which consists of constructing laminates  $\nu_j$  supported in

$$\bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} (A_j^i \cup B_i^j \cup S_{i,j}^a \cup S_{j,i}^a),$$

and then approximate those laminates by homeomorphisms using Proposition 18.

Given  $A \in \mathbb{R}^{n \times n}$ , let  $\sigma_1 \leq \dots \leq \sigma_n$  be its singular values. For  $i, k \in \mathbb{N} \setminus \{0\}$  we define the following sets in the case  $m_1 + m_2 \leq n - 1$ :

$$\begin{aligned}
A_k^i &= \left\{ A \in \Gamma_+ : |\sigma_j - (k+1)^{-1}| < \frac{(k+1)^{-2}}{4} \text{ for } j \in \{1, \dots, n-m_1\} \text{ and } \right. \\
&\quad \left. i - \frac{1}{4} < \sigma_j < i + \frac{5}{4} \text{ for } j \in \{n-m_1+1, \dots, n\} \right\}, \\
B_k^i &= \left\{ A \in \Gamma_+ : (k+1)^{-1} - \frac{(k+1)^{-2}}{4} < \sigma_j < k^{-1} + \frac{k^{-2}}{4} \text{ for } j \in \{1, \dots, m_2\} \right. \\
&\quad \left. \text{and } |\sigma_j - i - 1| < \frac{1}{4} \text{ for } j \in \{m_2+1, \dots, n\} \right\},
\end{aligned}$$

and for  $a \in \{m_2+1, \dots, n-m_1-1\}$  we define

$$S_{k,i}^a = \left\{ A \in \Gamma_+ : |\sigma_j - (k+1)^{-1}| < \frac{(k+1)^{-2}}{4} \text{ for } j \in \{1, \dots, a\} \text{ and } \right. \\
\left. |\sigma_j - i - 1| < \frac{1}{4} \text{ for } j \in \{a+1, \dots, n\} \right\}.$$

We will only prove the theorem in the previous case, since, in the case  $m_1 + m_2 \geq n$ , the proof is the same using the next sets instead of the above:

$$A_k^i = \left\{ A \in \Gamma_+ : |\sigma_j - (k+1)^{-1}| < \frac{(k+1)^{-2}}{4} \text{ for } j \in \{1, \dots, n-m_1\}, \frac{1}{2} < \sigma_j < 2 \right. \\
\left. \text{for } j \in \{n-m_1+1, \dots, m_2+1\} \text{ and } i - \frac{1}{4} < \sigma_j < i + \frac{5}{4} \text{ for } j \in \{m_2+2, \dots, n\} \right\},$$

and

$$B_k^i = \left\{ A \in \Gamma_+ : (k+1)^{-1} - \frac{(k+1)^{-2}}{4} < \sigma_j < k^{-1} + \frac{k^{-2}}{4} \text{ for } j \in \{1, \dots, n-m_1-1\}, \right. \\ \left. \frac{1}{2} < \sigma_j < 2 \text{ for } j \in \{n-m_1, \dots, m_2\} \text{ and } |\sigma_j - i - 1| < \frac{1}{4} \text{ for } j \in \{m_2+1, \dots, n\} \right\}.$$

The most important case is when  $m_1 + m_2 = n - 1$ , where we have  $S_{k,i}^a = \emptyset$ , and, therefore, the proof is much simpler. When  $m_1 + m_2 < n - 1$  the sets  $S_{k,i}^a$  constitute an interpolation between  $A_k^i$  and  $B_k^i$ . We recommend the reader to focus on the case  $m_1 + m_2 = n - 1$  in a first read.

In order to write all the lemmas in a form that include all the cases, we recall the definition

$$m'_i = \begin{cases} m_i & \text{if } m_1 + m_2 \leq n - 1, \\ m_i - (m_1 + m_2 - n + 1) & \text{if } m_1 + m_2 \geq n, \end{cases}$$

for  $i \in \{1, 2\}$ , and we define

$$n' = \begin{cases} n & \text{if } m_1 + m_2 \leq n - 1, \\ 2n - m_1 - m_2 - 1 & \text{if } m_1 + m_2 \geq n. \end{cases}$$

In the next lemma we construct a laminate supported in  $A_{k+1}^i \cup B_{k+1}^i \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{k+1,i}^a$  whose barycenter is a given matrix in  $A_k^i$ . We provide the proof in the case  $m_1 + m_2 \leq n - 1$ . If  $m_1 + m_2 \geq n$ , we fix the eigenvalues  $\{\sigma_j\}_{j=n-m_1+1}^{m_2+1}$ , which are the eigenvalues in  $(\frac{1}{2}, 2)$ , and we construct the same laminate as in the first case over the other eigenvalues, i.e., given  $A = \text{diag}(\sigma_1, \dots, \sigma_n) \in A_k^i$ , let  $A' = \text{diag}(\sigma_1, \dots, \sigma_{n-m_1}, \sigma_{m_2+2}, \dots, \sigma_n) \in \mathbb{R}^{n' \times n'}$  and apply Lemma 13 with  $n', m'_1$  and  $m'_2$  to get the laminate  $\nu = \sum_{\ell=1}^N \lambda_\ell \delta_{M'_\ell}$ , where

$$M'_\ell = \text{diag}(s_1^\ell, \dots, s_{n'}^\ell).$$

For  $\ell = 1, \dots, N$ , define

$$M_\ell = \text{diag}(s_1^\ell, \dots, s_{n-m_1}^\ell, \sigma_{n-m_1+1}(A), \dots, \sigma_{m_2+1}(A), s_{n-m_1+1}^\ell, \dots, s_{n'}^\ell).$$

So, in the case  $m_1 + m_2 \geq n$  we would work with the laminate  $\sum_{\ell=1}^N \lambda_\ell \delta_{M_\ell}$ .

The bounds of  $S_{k,i}^a$  along the paper only make sense when  $m_1 + m_2 < n$ ; otherwise, the  $S_{k,i}^a$  are empty.

**Lemma 13.** *Let  $A \in A_k^i$ . Then there exists a laminate of finite order  $\nu = \sum_{\ell=1}^N \lambda_\ell \delta_{M_\ell}$  such that*

- a)  $\bar{\nu} = A$ ,
- b)  $\text{supp}(\nu) \subset A_{k+1}^i \cup B_{k+1}^i \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{k+1,i}^a$ ,
- c)  $\nu(A_{k+1}^i) \leq 1$ ,



$$d) \nu(B_{k+1}^i) \lesssim (k^2 i)^{m'_1 + m'_2 - n'},$$

$$e) \nu(S_{k+1,i}^a) \lesssim (k^2 i)^{m'_1 + a - n'} \text{ for } a \in \{m_2 + 1, \dots, n - m_1 - 1\}.$$

$$f) M_1 \in A_{k+1}^i, |A - M_1| \leq k^{-2}, |A^{-1} - M_1^{-1}| \lesssim 1 \text{ and } 1 - \lambda_1 \lesssim k^{-2} i^{-1}.$$

*Proof.* Since we give the proof in the case  $m_1 + m_2 \leq n - 1$ , we have  $m'_1 = m_1$  and  $m'_2 = m_2$ . Without loss of generality we can assume that  $A$  is a diagonal matrix, hence  $A = \text{diag}(\sigma_1, \dots, \sigma_n)$  with  $|\sigma_j - (k+1)^{-1}| < \frac{(k+1)^{-2}}{4}$  for  $j \in \{1, \dots, n - m_1\}$  and  $i - \frac{1}{4} < \sigma_j < i + \frac{5}{4}$  for  $j \in \{n - m_1 + 1, \dots, n\}$ .

Let  $b = \#\{j \in \{n - m_1 + 1, \dots, n\} : \sigma_j > i + \frac{3}{4}\}$ .

We shall construct a family  $\{B_{\ell,j}\}_{\substack{\ell=0,\dots,n-b \\ j=0,\dots,2^\ell-1}}$  in  $\Gamma_+$  and a family  $\{\lambda_{\ell,j}\}_{\substack{\ell=0,\dots,n-b \\ j=0,\dots,2^\ell-1}}$  in  $[0, 1]$

by finite induction on  $\ell$ .

Let  $B_{0,0} = A$ ,  $\lambda_{0,0} = 1$  and for  $0 \leq \ell \leq n - b - 1$ ,  $0 \leq j \leq 2^\ell - 1$ , we assume that  $\{B_{\ell,j}\}_{j=0}^{2^\ell-1}$  and  $\{\lambda_{\ell,j}\}_{j=0}^{2^\ell-1}$  have been defined,  $B_{\ell,j}$  are diagonal,  $\lambda_{\ell,j} \geq 0$ ,

$$(8) \quad \sum_{j=0}^{2^\ell-1} \lambda_{\ell,j} = 1, \quad B_{0,0} = \sum_{j=0}^{2^\ell-1} \lambda_{\ell,j} B_{\ell,j}$$

$$(9) \quad \sum_{j=0}^{2^\ell-1} \lambda_{\ell,j} \delta_{B_{\ell,j}} \in \mathcal{L}(\mathbb{R}^{n \times n}),$$

and when we let

$$\begin{aligned} \beta_{\ell,j}^1 &:= \#\left\{\alpha \in \{1, \dots, n - m_1\} : |(B_{\ell,j})_{\alpha,\alpha} - (k+1)^{-1}| < \frac{(k+1)^{-2}}{4}\right\}, \\ \beta_{\ell,j}^2 &:= \#\left\{\alpha \in \{1, \dots, n - m_1\} : |(B_{\ell,j})_{\alpha,\alpha} - (k+2)^{-1}| < \frac{(k+2)^{-2}}{4}\right\}, \\ \beta_{\ell,j}^3 &:= \#\left\{\alpha \in \{1, \dots, n - m_1\} : |(B_{\ell,j})_{\alpha,\alpha} - i - 1| < \frac{1}{4}\right\}, \\ \gamma_{\ell,j}^1 &:= \#\left\{\alpha \in \{n - m_1 + 1, \dots, n - b\} : i - \frac{1}{4} < (B_{\ell,j})_{\alpha,\alpha} \leq i + \frac{3}{4}\right\}, \\ \gamma_{\ell,j}^2 &:= \#\left\{\alpha \in \{n - m_1 + 1, \dots, n - b\} : |(B_{\ell,j})_{\alpha,\alpha} - (k+2)^{-1}| < \frac{(k+2)^{-2}}{4}\right\}, \\ \gamma_{\ell,j}^3 &:= \#\left\{\alpha \in \{n - m_1 + 1, \dots, n - b\} : |(B_{\ell,j})_{\alpha,\alpha} - i - 1| < \frac{1}{4}\right\}, \end{aligned}$$

then

$$(10) \quad \beta_{\ell,j}^1 + \beta_{\ell,j}^2 + \gamma_{\ell,j}^2 \leq n - m_1,$$

$$(11) \quad \beta_{\ell,j}^3 + \gamma_{\ell,j}^1 + \gamma_{\ell,j}^3 \leq n - m_2 - b,$$

$$(12) \quad \beta_{\ell,j}^1 + \beta_{\ell,j}^2 + \beta_{\ell,j}^3 + \gamma_{\ell,j}^1 + \gamma_{\ell,j}^2 + \gamma_{\ell,j}^3 = n - b,$$

$$(13) \quad \lambda_{\ell,j} \leq \left( \frac{2}{i(k+1)^2} \right)^{\beta_{\ell,j}^3} \left( \frac{2}{i} \right)^{\gamma_{\ell,j}^2},$$

$$(14) \quad B_{\ell,0} = \text{diag} \left( \underbrace{(k+2)^{-1}, \dots, (k+2)^{-1}}_{\min\{\ell, n-m_1\}}, \sigma_{\min\{\ell, n-m_1\}+1}, \dots, \sigma_n \right)$$

and

$$(15) \quad \lambda_{\ell,0} = \prod_{j=1}^{\min\{\ell, n-m_1\}} \frac{i+1-\sigma_j}{i+1-(k+2)^{-1}}.$$

Moreover, for those  $B_{\ell,j} \notin A_{k+1}^i \cup B_{k+1}^i$ , we have

$$(16) \quad \beta_{\ell,j}^1 + \gamma_{\ell,j}^1 \leq n - \ell - b.$$

Since (12) holds and  $m_1 + m_2 \leq n - 1$ , we see that (10) and (11) cannot be equalities at the same time.

We observe that the sets appearing in the definitions of  $\beta_{\ell,j}^a, \gamma_{\ell,j}^a$  for  $a = 1, 2, 3$  are pairwise disjoint; we also see that

$$\beta_{0,0}^2 = \gamma_{0,0}^2 = \beta_{0,0}^3 = \gamma_{0,0}^3 = 0, \quad \beta_{0,0}^1 = n - m_1, \quad \gamma_{0,0}^1 = m_1 - b.$$

Now, we start with the induction. If  $\beta_{\ell,j}^2 + \gamma_{\ell,j}^2 = n - m_1$ , then  $B_{\ell,j} \in A_{k+1}^i$ , if  $\beta_{\ell,j}^3 + \gamma_{\ell,j}^3 = n - m_2 - b$ , then  $B_{\ell,j} \in B_{k+1}^i$ , and, if  $a := \beta_{\ell,j}^2 + \gamma_{\ell,j}^2 < n - m_1$ ,  $\beta_{\ell,j}^3 + \gamma_{\ell,j}^3 < n - m_2 - b$  and  $\beta_{\ell,j}^1 + \gamma_{\ell,j}^1 = 0$  then  $B_{\ell,j} \in S_{k+1,i}^a$ .

Now, for  $j = 0, \dots, 2^{\ell+1} - 1$  we construct  $B_{\ell+1,j}$  and  $\lambda_{\ell+1,j}$ .

If  $B_{\ell, \lfloor \frac{j}{2} \rfloor} \in A_{k+1}^i \cup B_{k+1}^i \cup \bigcup_{a=n-m_1+1}^{m_2-1} S_{k+1,i}^a$  we define  $B_{\ell+1,j} = B_{\ell, \lfloor \frac{j}{2} \rfloor}$  and

$$\lambda_{\ell+1,j} = \begin{cases} \lambda_{\ell, \frac{j}{2}}, & \text{if } j \text{ is even,} \\ 0, & \text{if } j \text{ is odd.} \end{cases}$$

So it is clear that (10)–(13) are satisfied.

In the case  $B_{\ell, \lfloor \frac{j}{2} \rfloor} \notin A_{k+1}^i \cup B_{k+1}^i \cup \bigcup_{a=n-m_1+1}^{m_2-1} S_{k+1,i}^a$ , we have

$$(17) \quad \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^2 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^2 < n - m_1,$$

$$(18) \quad \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^3 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^3 < n - m_2 - b,$$

and

$$(19) \quad \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^1 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^1 > 0,$$

and, we divide the construction of  $B_{\ell+1,j}$  into two cases, according to whether (20) or (21) holds. We observe that if  $B_{\ell,0} \in A_{k+1}^i \cup B_{k+1}^i \cup \bigcup_{a=n-m_1+1}^{m_2-1} S_{k+1,i}^a$  then  $B_{\ell,0} \in A_{k+1}^i$ , which happens if and only if  $\ell < n - m_1$ . Hence, if  $\ell \geq n - m_1$ , (14) and (15) are satisfied, whereas if  $\ell < n - m_1$  then (20) is satisfied for  $j = 0$ .

If

$$(20) \quad \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^1 > 0 \text{ and } \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^3 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^1 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^3 < n - m_2 - b,$$

let  $\alpha \in \{1, \dots, n - m_1\}$  be the smallest number such that  $|(B_{\ell, \lfloor \frac{j}{2} \rfloor})_{\alpha, \alpha} - (k+1)^{-1}| < \frac{(k+1)^{-2}}{4}$ . Then, we define

$$B_{\ell+1,j} = \begin{cases} B_{\ell, \lfloor \frac{j}{2} \rfloor} + \text{diag} \left( \underbrace{0, \dots, 0}_{\alpha-1}, (k+2)^{-1} - (B_{\ell, \lfloor \frac{j}{2} \rfloor})_{\alpha, \alpha}, \underbrace{0, \dots, 0}_{n-\alpha} \right), & \text{if } j \text{ is even,} \\ B_{\ell, \lfloor \frac{j}{2} \rfloor} + \text{diag} \left( \underbrace{0, \dots, 0}_{\alpha-1}, i+1 - (B_{\ell, \lfloor \frac{j}{2} \rfloor})_{\alpha, \alpha}, \underbrace{0, \dots, 0}_{n-\alpha} \right), & \text{if } j \text{ is odd,} \end{cases}$$

and

$$\lambda_{\ell+1,j} = \begin{cases} \frac{i+1 - (B_{\ell, \lfloor \frac{j}{2} \rfloor})_{\alpha, \alpha}}{i+1 - (k+2)^{-1}} \lambda_{\ell, \lfloor \frac{j}{2} \rfloor} \leq \lambda_{\ell, \lfloor \frac{j}{2} \rfloor}, & \text{if } j \text{ is even,} \\ \frac{(B_{\ell, \lfloor \frac{j}{2} \rfloor})_{\alpha, \alpha} - (k+2)^{-1}}{i+1 - (k+2)^{-1}} \lambda_{\ell, \lfloor \frac{j}{2} \rfloor} \leq \frac{2}{i(k+1)^2} \lambda_{\ell, \lfloor \frac{j}{2} \rfloor}, & \text{if } j \text{ is odd.} \end{cases}$$

Then

$$B_{\ell, \frac{j}{2}} = \frac{i+1 - (B_{\ell, \frac{j}{2}})_{\alpha, \alpha}}{i+1 - (k+2)^{-1}} B_{\ell+1,j} + \frac{(B_{\ell, \frac{j}{2}})_{\alpha, \alpha} - (k+2)^{-1}}{i+1 - (k+2)^{-1}} B_{\ell+1,j+1} \text{ for } j \in \{0, \dots, 2^{\ell+1}-1\} \text{ even,}$$

and hence

$$\begin{aligned} & \sum_{j=0}^{2^{\ell+1}-1} \lambda_{\ell+1,j} \delta_{B_{\ell+1,j}} \in \mathcal{L}(\mathbb{R}^{n \times n}), \\ & \beta_{\ell+1,j}^1 = \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^1 - 1, \\ & \beta_{\ell+1,j}^2 = \begin{cases} \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^2 + 1, & \text{if } j \text{ is even,} \\ \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^2, & \text{if } j \text{ is odd,} \end{cases} \end{aligned}$$

$$\beta_{\ell+1,j}^3 = \begin{cases} \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^3, & \text{if } j \text{ is even,} \\ \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^3 + 1, & \text{if } j \text{ is odd,} \end{cases}$$

$$\gamma_{\ell+1,j}^1 = \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^1, \quad \gamma_{\ell+1,j}^2 = \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^2 \quad \text{and} \quad \gamma_{\ell+1,j}^3 = \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^3.$$

Therefore, (8)–(16) are satisfied for  $\ell + 1$ .

If

$$(21) \quad \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^1 = 0 \text{ or } \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^3 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^1 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^3 = n - m_2 - b$$

instead of (20), we claim that then we have  $\gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^1 > 0$  and

$$\beta_{\ell, \lfloor \frac{j}{2} \rfloor}^1 + \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^2 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^2 < n - m_1.$$

Indeed, if  $\beta_{\ell, \lfloor \frac{j}{2} \rfloor}^1 = 0$ , it is clear thanks to (17) and (19), and if

$$\beta_{\ell, \lfloor \frac{j}{2} \rfloor}^3 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^1 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^3 = n - m_2 - b,$$

we have

$$\beta_{\ell, \lfloor \frac{j}{2} \rfloor}^1 + \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^2 + \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^2 = m_2 < n - m_1,$$

and by (18) we obtain  $\gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^1 > 0$ . Therefore there exists  $\alpha \in \{n - m_1 + 1, \dots, n - b\}$  such that

$$i - \frac{1}{4} < (B_{\ell, \lfloor \frac{j}{2} \rfloor})_{\alpha, \alpha} \leq i + \frac{3}{4}.$$

Then, we define

$$B_{\ell+1,j} = \begin{cases} B_{\ell, \lfloor \frac{j}{2} \rfloor} + \text{diag} \left( \underbrace{0, \dots, 0}_{\alpha-1}, (k+2)^{-1} - (B_{\ell, \lfloor \frac{j}{2} \rfloor})_{\alpha, \alpha}, \underbrace{0, \dots, 0}_{n-\alpha} \right), & \text{if } j \text{ is even,} \\ B_{\ell, \lfloor \frac{j}{2} \rfloor} + \text{diag} \left( \underbrace{0, \dots, 0}_{\alpha-1}, i+1 - (B_{\ell, \lfloor \frac{j}{2} \rfloor})_{\alpha, \alpha}, \underbrace{0, \dots, 0}_{n-\alpha} \right), & \text{if } j \text{ is odd,} \end{cases}$$

$$\lambda_{\ell+1,j} = \begin{cases} \frac{i+1-(B_{\ell, \lfloor \frac{j}{2} \rfloor})_{\alpha, \alpha}}{i+1-(k+2)^{-1}} \lambda_{\ell, \lfloor \frac{j}{2} \rfloor} \leq \frac{2}{i} \lambda_{\ell, \lfloor \frac{j}{2} \rfloor}, & \text{if } j \text{ is even,} \\ \frac{(B_{\ell, \lfloor \frac{j}{2} \rfloor})_{\alpha, \alpha} - (k+2)^{-1}}{i+1-(k+2)^{-1}} \lambda_{\ell, \lfloor \frac{j}{2} \rfloor} \leq \lambda_{\ell, \lfloor \frac{j}{2} \rfloor}, & \text{if } j \text{ is odd.} \end{cases}$$

Then

$$\sum_{j=0}^{2^{\ell+1}-1} \lambda_{\ell+1,j} \delta_{B_{\ell+1,j}} \in \mathcal{L}(\mathbb{R}^{n \times n}),$$

$$\beta_{\ell+1,j}^1 = \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^1, \quad \beta_{\ell+1,j}^2 = \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^2, \quad \beta_{\ell+1,j}^3 = \beta_{\ell, \lfloor \frac{j}{2} \rfloor}^3,$$

$$\gamma_{\ell+1,j}^1 = \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^1 - 1,$$

$$\gamma_{\ell+1,j}^2 = \begin{cases} \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^2 + 1, & \text{if } j \text{ is even,} \\ \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^2, & \text{if } j \text{ is odd,} \end{cases}$$

and

$$\gamma_{\ell+1,j}^3 = \begin{cases} \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^3, & \text{if } j \text{ is even,} \\ \gamma_{\ell, \lfloor \frac{j}{2} \rfloor}^3 + 1, & \text{if } j \text{ is odd.} \end{cases}$$

Therefore, (8), (9), (10), (11), (12), (13) and (16) are satisfied for  $\ell + 1$ .

Here ends the inductive construction of  $\{B_{\ell,j}\}_{\ell=0,\dots,n-b, j=0,\dots,2^\ell-1}$ . With this, we define

$$N = 2^{n-b}, \quad \lambda_\ell = \lambda_{n-b,\ell-1}, \quad M_\ell = B_{n-b,\ell-1} \text{ for } \ell = 1, \dots, 2^{n-b} \quad \text{and } \nu := \sum_{j=1}^{2^{n-b}} \lambda_j \delta_{M_j},$$

which is a laminate by (9), and by (12) and (16) it is supported in  $A_{k+1}^i \cup B_{k+1}^i \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{k+1,i}^a$ . We shall check properties *a)*–*f)*. Property *a)* comes from (8), *b)* is by (16), and *c)* is obvious.

Now, we use (13) to bound the mass of  $\nu$  in the different sets.

Since  $\gamma_{n-b,j}^3 \leq m_1 - b$  and the matrices  $B_{n-b,j}$  in  $B_{k+1}^i$  are those such that  $\beta_{n-b,j}^3 + \gamma_{n-b,j}^3 = n - m_2 - b$ , we have  $\beta_{n-b,j}^3 \geq n - m_1 - m_2$  and

$$\nu(B_{k+1}^i) = \sum_{j: \beta_{n-b,j}^3 + \gamma_{n-b,j}^3 = n - m_2 - b} \lambda_{n-b,j} \lesssim \left( \frac{2}{i(k+1)^2} \right)^{n-m_1-m_2} \lesssim (k^2 i)^{m_1+m_2-n}.$$

So *d)* is proved. Now we use that  $\sum_{l=1}^3 \beta_{n-b,j}^l = n - m_1$  and that for  $a \in \{m_2+1, \dots, n-m_1-1\}$  the matrices  $B_{n-b,j}$  in  $S_{k+1,i}^a$  are those such that  $\beta_{n-b,j}^2 + \gamma_{n-b,j}^2 = a$ ,  $\beta_{n-b,j}^1 = \gamma_{n-b,j}^1 = 0$ , so

$$\beta_{n-b,j}^3 = n - m_1 - \beta_{n-b,j}^2 = n - m_1 + \gamma_{n-b,j}^2 - a \geq n - m_1 - a,$$

and, hence,

$$\nu(S_{k+1,i}^a) = \sum_{j: \beta_{n-b,j}^2 + \gamma_{n-b,j}^2 = a} \lambda_{n-b,j} \lesssim \left( \frac{2}{i(k+1)^2} \right)^{n-m_1-a} \lesssim (k^2 i)^{m_1+a-n}.$$

Therefore we have *e)*. Finally, we prove *f)*. Thanks to (14) we get

$$M_1 = \text{diag} \left( \underbrace{(k+2)^{-1}, \dots, (k+2)^{-1}}_{n-m_1}, \sigma_{n-m_1+1}, \dots, \sigma_n \right)$$

and due to (15) we obtain

$$\lambda_1 = \prod_{j=1}^{n-m_1} \frac{i+1-\sigma_j}{i+1-(k+2)^{-1}}.$$

Therefore, using  $k^{-1} - (k+1)^{-1} \leq k^{-2}$  we get

$$|A - M_1| \leq k^{-2}, \quad |A^{-1} - M^{-1}| \lesssim 1$$

and, noting that  $n - m_1 \geq 2$  we also obtain

$$\begin{aligned} 1 - \lambda_1 &\leq 1 - \left( \frac{i+1 - (k+1)^{-1} - \frac{(k+1)^{-2}}{4}}{i+1 - (k+2)^{-1}} \right)^{n-m_1} \\ &\lesssim \frac{\left( (k+1)^{-1} + \frac{(k+1)^{-2}}{4} \right)^{n-m_1-1} - (k+2)^{m_1-n+1}}{i} \lesssim k^{-2} i^{-1}. \end{aligned}$$

Hence, the proof is finished.  $\square$

In the next two lemmas we give the laminates starting in  $B_k^i$  and  $S_{k,i}^a$ . We will not show them, since their construction mimic that of Lemma 13. The main difference in the proofs of Lemmas 14 and 15 with the one of Lemma 13 is that we split the eigenvalues  $\sigma$  close to  $i+1$  in  $(k+1)^{-1}$  and  $\sigma+1$ . Also, in the proof of Lemma 14 we start splitting the eigenvalues close to  $i+1$  and we do not split the eigenvalues in  $\left( (k+1)^{-1} - \frac{(k+1)^{-2}}{4}, (k+1)^{-1} + \frac{(k+1)^{-2}}{4} \right)$ .

**Lemma 14.** *Let  $A \in B_k^i$ . Then there exists a laminate of finite order  $\nu = \sum_{\ell=1}^N \lambda_\ell \delta_{M_\ell}$  such that*

- $\bar{\nu} = A$ ,
- $\text{supp}(\nu) \subset A_k^{i+1} \cup B_k^{i+1} \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{k,i+1}^a$ ,
- $\nu(A_k^{i+1}) \lesssim i^{m'_1+m'_2-n'}$ ,
- $\nu(B_k^{i+1}) - \left( \frac{i+2}{i+3} \right)^{n'-m'_2} \lesssim (ki)^{-2}$ ,
- $\nu(S_{k,i+1}^a) \lesssim i^{m'_2-a}$  for  $a \in \{m_2+1, \dots, n-m_1-1\}$ ,
- $M_1 \in B_k^{i+1}$ ,  $|A - M_1| = 1$ ,  $|A^{-1} - M_1^{-1}| \leq i^{-2}$  and  $1 - \lambda_1 \lesssim i^{-1}$ .

**Lemma 15.** *Let  $a_0 \in \{m_2+1, \dots, n-m_1-1\}$  and  $A \in S_{k,i}^{a_0}$ . Then there exists a laminate of finite order  $\nu = \sum_{\ell=1}^N \lambda_\ell \delta_{M_\ell}$  such that*

- $\bar{\nu} = A$ ,
- $\text{supp}(\nu) \subset A_{k+1}^{i+1} \cup B_{k+1}^{i+1} \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{k+1,i+1}^a$ ,
- $\nu(A_{k+1}^{i+1}) \lesssim i^{a_0+m_1-n}$ ,
- $\nu(B_{k+1}^{i+1}) \lesssim (k^2 i)^{m_2-a_0}$ ,

- $\nu(S_{k+1,i+1}^a) \lesssim (k^2 i)^{a-a_0}$  if  $a \in \{m_2 + 1, \dots, a_0 - 1\}$ ,
- $\nu(S_{k+1,i+1}^{a_0}) - \left(\frac{i+2}{i+3}\right)^{n-a_0} \lesssim (ki)^{-2}$ ,
- $\nu(S_{k+1,i+1}^a) \lesssim i^{a_0-a}$  if  $a \in \{a_0 + 1, \dots, n - m_1 - 1\}$ ,
- $\sum_{\ell=1}^N \lambda_\ell |A - M_\ell| \lesssim 1$ ,
- $\frac{1}{\det(A)} \sum_{\ell=1}^N \lambda_\ell \det(M_\ell) |A^{-1} - M_\ell^{-1}| \lesssim 1$ .

In the proof of the last lemma, besides adapting the proof of Lemma 13, we follow the proof of  $h)$  and  $i)$  of Lemma 16 below to show the last two items.

In the next lemma we put together Lemmas 13, 14 and 15 to construct laminates whose support is in the set in which we are interested. Again, all the bounds of  $S_{j+1,i}^a$  have sense if and only if  $m_1 + m_2 \leq n - 1$ ; otherwise these sets are empty.

**Lemma 16.** *Let  $i, j \in \mathbb{N}$ ,  $i \leq j$ , and  $A \in A_j^i$ . Then there exists a laminate  $\nu = \sum_{k=1}^N \lambda_k \delta_{M_k}$  such that*

- a)  $\bar{\nu} = A$ ,
- b)  $\text{supp}(\nu) \subset \bigcup_{b=0}^{j-i+1} A_{j+1}^{i+b} \cup B_{j+1}^{j+1} \cup \bigcup_{b=0}^{j-i+1} \bigcup_{a=m_2+1}^{n-m_1-1} S_{j+1,i+b}^a$ ,
- c)  $\nu(A_{j+1}^i) \leq 1$ ,
- d)  $\nu(A_{j+1}^{i+b}) \lesssim j^{2(m'_1+m'_2-n')} \frac{i^{m'_1}}{(i+b)^{m'_1+2}}$ , for  $b \in \{1, \dots, j - i + 1\}$ ,
- e)  $\nu(B_{j+1}^{j+1}) \lesssim j^{2m'_1+3m'_2-3n'} i^{m'_1}$ ,
- f)  $\nu(S_{j+1,i}^a) \lesssim (j^2 i)^{m'_1+a-n'}$ ,
- g)  $\nu(S_{j+1,i+b}^a) \lesssim j^{2(m'_1+m'_2-n')} i^{m'_1} (i+b)^{2m'_2-a-n'}$  for  $a \in \{m_2 + 1, \dots, n - m_1 - 1\}$  and  $b \in \{1, \dots, j - i + 1\}$ ,
- h)  $\sum_{k=1}^N \lambda_k |A - M_k| \lesssim j^{-2}$ ,
- i)  $\frac{1}{\det(A)} \sum_{k=1}^N \lambda_k \det(M_k) |A^{-1} - M_k^{-1}| \lesssim 1$ .

*Proof.* Let  $C$  be a constant bigger than those in Lemma 14. It is enough construct a sequence of laminates  $\{\nu_\ell\}_{\ell=1}^{j-i+2}$  such that

- 1)  $\bar{\nu}_\ell = A$ ,
- 2)  $\text{supp}(\nu_\ell) \subset \bigcup_{b=0}^{\ell-1} A_{j+1}^{i+b} \cup B_{j+1}^{i+\ell-1} \cup \bigcup_{b=0}^{\ell-1} \bigcup_{a=m_2+1}^{n-m_1-1} S_{j+1,i+b}^a$ ,
- 3)  $\nu_\ell(A_{j+1}^{i+b}) \leq j^{2(m'_1+m'_2-n')} \frac{i^{m'_1}}{(i+b)^{m'_1+2}}$ , for  $b \in \{1, \dots, \ell - 1\}$ ,

- 4)  $\nu_\ell(B_{j+1}^{i+\ell-1}) \lesssim j^{2(m'_1+m'_2-n')} i^{m'_1} (i+\ell+1)^{m'_2-n'} \left(1 + j^{-2} \sum_{k=1}^{\ell-1} 2^n C(i+k)^{-2}\right),$
- 5)  $\nu_\ell(S_{j+1,i}^a) \lesssim (j^2 i)^{m'_1+a-n'},$
- 6)  $\nu_\ell(S_{j+1,i+b}^a) \lesssim j^{2(m'_1+m'_2-n')} i^{m'_1} (i+b)^{2m'_2-a-n'}$  for  $a \in \{m_2+1, \dots, n-m_1-1\}$  and  $b \in \{1, \dots, \ell-1\},$
- 7)  $M_1 \in A_{j+1}^i \cap \text{supp}(\nu_\ell)$  such that  $|A - M_1| \leq j^{-2}, |A^{-1} - M_1^{-1}| \lesssim 1, 1 - \nu_\ell(M_1) \lesssim j^{-2} i^{-1},$  with  $M_1$  being the one of Lemma 13,

and prove later  $h)$  and  $i)$ .

Let  $\nu_1$  be the laminate of Lemma 13, which satisfies

- $\bar{\nu}_1 = A,$
- $\text{supp}(\nu_1) \subset A_{j+1}^i \cup B_{j+1}^i \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{j+1,i}^a,$
- $\nu_1(B_{j+1}^i) \lesssim (j^2 i)^{m'_1+m'_2-n'},$
- $\nu_1(S_{j+1,i}^a) \lesssim (j^2 i)^{m'_1+a-n'}$  for  $a \in \{m_2+1, \dots, n-m_1-1\},$
- $\exists M \in A_{j+1}^i \cap \text{supp}(\nu_1)$  such that  $|A - M| \leq j^{-2}, |A^{-1} - M^{-1}| \lesssim 1$  and  $1 - \nu_1(M) \lesssim j^{-2} i^{-1}.$

Therefore, denoting by  $M_1$  the matrix  $M$ , we get that  $\nu_1$  satisfies 1)–7).

Now, we proceed by induction and assume that  $\nu_\ell$  has been constructed with the properties 1)–7). For each  $B \in \text{supp}(\nu_\ell) \cap B_{j+1}^{i+\ell-1}$ , let  $\nu_B$  be the laminate given by Lemma 14, which satisfies

- $\bar{\nu}_B = B,$
- $\text{supp}(\nu_B) \subset A_{j+1}^{i+\ell} \cup B_{j+1}^{i+\ell} \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{j+1,i+\ell}^a,$
- $\nu_B(A_{j+1}^{i+\ell}) \lesssim (i+\ell)^{m'_1+m'_2-n'},$
- $\nu_B(B_{j+1}^{i+\ell}) - \left(\frac{i+\ell+1}{i+\ell+2}\right)^{n'-m'_2} \leq C(i+\ell)^{-2} j^{-2},$
- $\nu_B(S_{j+1,i+\ell}^a) \lesssim (i+\ell)^{m'_2-a}$  for  $a \in \{m_2+1, \dots, n-m_1-1\}.$

We define

$$\nu_{\ell+1} = \nu_\ell + \sum_{B \in \text{supp}(\nu_\ell) \cap B_{j+1}^{i+\ell-1}} \nu_\ell(B)(\nu_B - \delta_B).$$

Thanks to Corollary 7,  $\nu_{\ell+1}$  is a laminate. Moreover, it is clear that  $\bar{\nu}_{\ell+1} = A$  and

$$\text{supp}(\nu_{\ell+1}) \subset \bigcup_{b=0}^{\ell} A_{j+1}^{i+b} \cup B_{j+1}^{i+\ell} \cup \bigcup_{b=0}^{\ell} \bigcup_{a=m_2+1}^{n-m_1-1} S_{j+1,i+b}^a.$$



Now, observe that the matrices in  $(A_{j+1}^{i+\ell} \cup B_{j+1}^{i+\ell} \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{j+1,i+\ell}^a) \cap \text{supp}(\nu_{\ell+1})$  are those in the support of  $\nu_B$  for  $B$  in  $B_{j+1}^{i+\ell-1} \cap \text{supp}(\nu_\ell)$ . Therefore, using  $m'_1 + m'_2 \leq n' - 1$  we have

$$\begin{aligned}
\nu_{\ell+1}(A_{j+1}^{i+\ell}) &= \sum_{B \in B_{j+1}^{i+\ell-1}} \nu_\ell(B) \nu_B(A_{j+1}^{i+\ell}) \leq \sum_{B \in B_{j+1}^{i+\ell-1}} \nu_\ell(B) (i+\ell)^{m'_1+m'_2-n'} \\
&= \nu_\ell(B_{j+1}^{i+\ell-1}) (i+\ell)^{m'_1+m'_2-n'} \\
&\lesssim j^{2(m'_1+m'_2-n')} i^{m'_1} (i+\ell+1)^{m'_2-n'} \left( 1 + j^{-2} \sum_{k=1}^{\ell-1} 2^n C(i+k)^{-2} \right) (i+\ell)^{m'_1+m'_2-n'} \\
&\leq j^{2(m'_1+m'_2-n')} i^{m'_1} (i+\ell+1)^{-m'_1-2} \left( 1 + j^{-2} \sum_{k=1}^{\ell-1} 2^n C(i+k)^{-2} \right),
\end{aligned}$$

Therefore

$$\nu_{\ell+1}(A_{j+1}^{i+\ell}) \lesssim j^{2(m'_1+m'_2-n')} \frac{i^{m'_1}}{(i+\ell)^{m'_1+2}}.$$

For the bound of  $\nu_{\ell+1}(B_{j+1}^{i+\ell})$  we need the following estimate, in which we use  $j \geq 2^n C$ :

$$\begin{aligned}
&\left( 1 + j^{-2} \sum_{k=1}^{\ell-1} 2^n C(i+k)^{-2} \right) \left( 1 + C \left( \frac{i+\ell+2}{i+\ell+1} \right)^{n'-m'_2} j^{-2} (i+\ell)^{-2} \right) \\
&\leq 1 + j^{-2} \sum_{k=1}^{\ell-1} 2^n C(i+k)^{-2} + 2C 2^{n-1} j^{-2} (i+\ell)^{-2} \leq 1 + j^{-2} \sum_{k=1}^{\ell} 2^n C(i+k)^{-2}.
\end{aligned}$$

Proceeding in the same way as in the bound of  $\nu_{\ell+1}(A_{j+1}^{i+\ell})$ , we obtain the following bound. Note that the constant corresponding to the symbol  $\lesssim$  is the same for all  $\ell = 1, \dots, j-i+2$ :

$$\begin{aligned}
\nu_{\ell+1}(B_{j+1}^{i+\ell}) &= \sum_{B \in B_{j+1}^{i+\ell-1}} \nu_\ell(B) \nu_B(B_{j+1}^{i+\ell}) \leq \sum_{B \in B_{j+1}^{i+\ell-1}} \nu_\ell(B) \left[ \left( \frac{i+\ell+1}{i+\ell+2} \right)^{n'-m'_2} + C(i+\ell)^{-2} j^{-2} \right] \\
&= \nu_\ell(B_{j+1}^{i+\ell-1}) \left[ \left( \frac{i+\ell+1}{i+\ell+2} \right)^{n'-m'_2} + C(i+\ell)^{-2} j^{-2} \right] \\
&\lesssim j^{2(m'_1+m'_2-n')} i^{m'_1} (i+\ell+1)^{m'_2-n'} \left( 1 + j^{-2} \sum_{k=1}^{\ell-1} 2^n C(i+k)^{-2} \right) \\
&\quad \times \left[ \left( \frac{i+\ell+1}{i+\ell+2} \right)^{n'-m'_2} + C(i+\ell)^{-2} j^{-2} \right] \\
&\leq j^{2(m'_1+m'_2-n')} i^{m'_1} (i+\ell+2)^{m'_2-n'} \left( 1 + j^{-2} \sum_{k=1}^{\ell} 2^n C(i+k)^{-2} \right).
\end{aligned}$$

Next we bound  $\nu_{\ell+1}(S_{j+1,i+\ell}^a)$  for  $a \in \{m_2 + 1, \dots, n - m_1 - 1\}$ :

$$\begin{aligned}
\nu_{\ell+1}(S_{j+1,i+\ell}^a) &= \sum_{B \in B_{j+1}^{i+\ell-1}} \nu_\ell(B) \nu_B(S_{j+1,i+\ell}^a) \leq \sum_{B \in B_{j+1}^{i+\ell-1}} \nu_\ell(B) (i + \ell)^{m'_2 - a} \\
&= \nu_\ell(B_{j+1}^{i+\ell-1}) (i + \ell)^{m'_2 - a} \\
&\lesssim j^{2(m'_1 + m'_2 - n')} i^{m'_1} (i + \ell + 1)^{m'_2 - n'} \left( 1 + j^{-2} \sum_{k=1}^{\ell-1} 2^n C(i+k)^{-2} \right) (i + \ell)^{m'_2 - a} \\
&\lesssim j^{2(m'_1 + m'_2 - n')} i^{m'_1} (i + \ell)^{2m'_2 - a - n'}.
\end{aligned}$$

For  $b \in \{0, \dots, \ell\}$ , we also have that

$$\nu_{\ell+1} \lfloor_{A_{j+1}^{i+b}} = \nu_\ell \lfloor_{A_{j+1}^{i+b}}$$

and

$$\nu_{\ell+1} \lfloor_{S_{j+1,i+b}^a} = \nu_\ell \lfloor_{S_{j+1,i+b}^a} \text{ for } a \in \{m_2 + 1, \dots, n - m_1 - 1\}.$$

Here,  $\lfloor$  denotes the restriction of a measure. Therefore,  $\nu_{\ell+1}$  satisfies 1)–7). Here ends the inductive construction of  $\{\nu_\ell\}_{\ell=1}^{j-i+2}$ .

Now, we define  $\nu = \nu_{j-i+2} = \sum_{k=1}^N \lambda_k \delta_{M_k}$ , so  $\lambda_1 = \nu_1(M_1)$  and we recall that

$$M_1 \in A_{j+1}^i \cap \text{supp}(\nu), |A - M_1| \leq j^{-2}, |A^{-1} - M_1^{-1}| \lesssim 1 \text{ and } 1 - \lambda_1 \lesssim j^{-2} i^{-1}.$$

Finally we have to prove *h)* and *i)*. To show *h)* we need the following estimate of the distance between the matrices in the support of  $\nu$  and  $A$ :

$$|A - M| \leq |A| + |M| \lesssim \begin{cases} i + b & \text{if } M \in A_{j+1}^{i+b} \setminus \{M_1\} \text{ for some } b \in \{0, \dots, j - i + 1\}, \\ j & \text{if } M \in B_{j+1}^{j+1}, \\ i + b & \text{if } M \in S_{j+1,i+b}^a \text{ for some } b \in \{0, \dots, j - i + 1\}. \end{cases}$$

Now we split the sum  $\sum_{k=1}^N \lambda_k |A - M_k|$  over the different sets and we bound those sums:

$$\lambda_1 |A - M_1| \lesssim j^{-2},$$

$$\sum_{k: M_k \in A_{j+1}^i \setminus \{M_1\}} \lambda_k |A - M_k| \lesssim \sum_{k: M_k \in A_{j+1}^i \setminus \{M_1\}} \lambda_k i \leq i(1 - \lambda_1) \lesssim j^{-2}.$$

In the following estimate we use  $\sum_{b=0}^\infty (i+b)^{-m_1-1} \lesssim i^{-m_1}$  and  $m'_1 + m'_2 - n' \leq -1$ :

$$\begin{aligned}
\sum_{b=1}^{j-i+1} \sum_{k: M_k \in A_{j+1}^{i+b}} \lambda_k |A - M_k| &\lesssim \sum_{b=1}^{j-i+1} \sum_{k: M_k \in A_{j+1}^{i+b}} \lambda_k (i + b) = \sum_{b=1}^{j-i+1} \nu(A_{j+1}^{i+b}) (i + b) \\
&\lesssim \sum_{b=1}^{j-i+1} j^{2(m'_1 + m'_2 - n')} \frac{i^{m'_1}}{(i + b)^{m'_1 + 1}} \lesssim j^{-2}.
\end{aligned}$$

Using that  $i \leq j$  and  $m'_1 + m'_2 \leq n' - 1$ , we have

$$\sum_{k: M_k \in B_{j+1}^{j+1}} \lambda_k |A - M_k| \lesssim \sum_{k: M_k \in B_{j+1}^{j+1}} \lambda_k j = \nu(B_{j+1}^{j+1}) j \lesssim j^{2m'_1 + 3m'_2 - 3n' + 1} i^{m'_1} \leq j^{-2}.$$

In the same way as before, we bound the sum over the sets  $S_{j+1,i}^a$  and  $S_{j+1,i}^{a+b}$ ; using that in this case  $m'_1 = m_1$  and  $n' = n$ , we get

$$\begin{aligned} \sum_{a=m_2+1}^{n-m_1-1} \sum_{k: M_k \in S_{j+1,i}^a} \lambda_k |A - M_k| &\lesssim \sum_{a=m_2+1}^{n-m_1-1} \sum_{k: M_k \in S_{j+1,i}^a} \lambda_k i = \sum_{a=m_2+1}^{n-m_1-1} \nu(S_{j+1,i}^a) i \\ &\lesssim \sum_{a=m_2+1}^{n-m_1-1} j^{2(m'_1 + a - n')} i^{m'_1 + a - n' + 1} \lesssim j^{-2} \end{aligned}$$

and

$$\begin{aligned} \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} \sum_{k: M_k \in S_{j+1,i+b}^a} \lambda_k |A - M_k| &\lesssim \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} \sum_{k: M_k \in S_{j+1,i+b}^a} \lambda_k (i+b) \\ &= \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} \nu(S_{j+1,i+b}^a) (i+b) \lesssim \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} j^{2(m'_1 + m'_2 - n')} i^{m'_1} (i+b)^{2m'_2 - a - n' + 1} \\ &\lesssim j^{-2} i^{m'_1} \sum_{b=1}^{j-i+1} (i+b)^{m'_2 - n'} \leq j^{-2} i^{m'_1} \sum_{b=1}^{j-i+1} (i+b)^{-m'_1 - 1} \lesssim j^{-2}. \end{aligned}$$

Therefore, putting together all the previous bounds we obtain

$$\sum_{k=1}^N \lambda_k |A - M_k| \lesssim j^{-2},$$

and, hence,  $h)$  is proved.

Now, to prove  $i)$  we need to bound the distance between the inverses. For  $M \in \text{supp}(\nu) \setminus \{M_1\}$  we have

$$|A^{-1} - M^{-1}| \leq |A^{-1}| + |M^{-1}| \lesssim j.$$

We also need the following bound of the determinants. Since  $A \in A_j^i$  and  $m_1 - n = m'_1 - n'$  we have

$$\det(A) \gtrsim j^{m_1 - n} i^{m'_1} = j^{m'_1 - n'} i^{m'_1}.$$

Looking at the definition of the sets we also get

$$\det(M) \lesssim \begin{cases} j^{m'_1 - n'} (i+b)^{m'_1} & \text{if } M \in A_{j+1}^{i+b} \text{ for some } b \in \{0, \dots, j-i+1\}, \\ j^{n' - 2m'_2} & \text{if } M \in B_{j+1}^{j+1}, \\ j^{-a} (i+b)^{n'-a} & \text{if } M \in S_{j+1,i+b}^a \text{ for some } b \in \{0, \dots, j-i+1\}. \end{cases}$$

Hence we get

$$\frac{\det(M)}{\det(A)} |A^{-1} - M^{-1}| \lesssim \begin{cases} j(i+b)^{m'_1} i^{-m'_1} & \text{if } M \in A_{j+1}^{i+b} \setminus \{M_1\} \text{ for some} \\ & b \in \{0, \dots, j-i+1\}, \\ j^{2n'-2m'_2-m'_1+1} i^{-m'_1} & \text{if } M \in B_{j+1}^{j+1}, \\ j^{n'-m'_1-a+1} (i+b)^{n'-a} i^{-m'_1} & \text{if } M \in S_{j+1,i+b}^a \text{ for some} \\ & b \in \{0, \dots, j-i+1\}. \end{cases}$$

Now we split the sum  $\sum_{k=1}^N \lambda_k \frac{\det(M_k)}{\det(A)} |A^{-1} - M_k^{-1}|$  over the different sets and bound those sums:

$$\lambda_1 \frac{\det(M_1)}{\det(A)} |A^{-1} - M_1^{-1}| \lesssim \frac{\det(M_1)}{\det(A)} \lesssim j^{m'_1-n'} i^{m'_1} j^{n'-m'_1} i^{-m'_1} = 1,$$

$$\sum_{k: M_k \in A_{j+1}^i \setminus \{M_1\}} \lambda_k \frac{\det(M_k)}{\det(A)} |A^{-1} - M_k^{-1}| \lesssim \sum_{k: M_k \in A_{j+1}^i \setminus \{M_1\}} \lambda_k j \leq j(1 - \lambda_1) \lesssim j^{-1} i^{-1} \leq 1.$$

In the following estimate we use  $\sum_{b=0}^\infty (i+b)^{-2} \lesssim i^{-1}$  and  $m'_1 + m'_2 - n' \leq -1$ :

$$\begin{aligned} \sum_{b=1}^{j-i+1} \sum_{k: M_k \in A_{j+1}^{i+b}} \lambda_k \frac{\det(M_k)}{\det(A)} |A^{-1} - M_k^{-1}| &\lesssim \sum_{b=1}^{j-i+1} \sum_{k: M_k \in A_{j+1}^{i+b}} \lambda_k j \frac{(i+b)^{m'_1}}{i^{m'_1}} \\ &\leq \sum_{b=1}^{j-i+1} \nu(A_{j+1}^{i+b}) j \frac{(i+b)^{m'_1}}{i^{m'_1}} = \sum_{b=1}^{j-i+1} j^{2(m'_1+m'_2-n')+1} \frac{i^{m'_1}}{(i+b)^{m'_1+2}} \frac{(i+b)^{m'_1}}{i^{m'_1}} \lesssim j^{-1} i^{-1} \leq 1. \end{aligned}$$

Using that  $i \leq j$  and  $m'_1 + m'_2 \leq n' - 1$ , we have

$$\begin{aligned} \sum_{k: M_k \in B_{j+1}^{j+1}} \lambda_k \frac{\det(M_k)}{\det(A)} |A^{-1} - M_k^{-1}| &\lesssim \sum_{k: M_k \in B_{j+1}^{j+1}} \lambda_k j^{2n'-2m'_2-m'_1+1} i^{-m'_1} \\ &= \nu(B_{j+1}^{j+1}) j^{2n'-2m'_2-m'_1+1} i^{-m'_1} \lesssim j^{2m'_1+3m'_2-3n'} i^{m'_1} j^{2n'-2m'_2-m'_1+1} i^{-m'_1} = j^{m'_1+m'_2-n'+1} \leq 1. \end{aligned}$$

In the same way as before we bound the sum over the sets  $S_{j+1,i}^a$  and  $S_{j+1,i}^{a+b}$ ; using that in this case we have  $m'_1 = m_1$  and  $n' = n$ , we get

$$\begin{aligned} \sum_{a=m_2+1}^{n-m_1-1} \sum_{k: M_k \in S_{j+1,i}^a} \lambda_k \frac{\det(M_k)}{\det(A)} |A^{-1} - M_k^{-1}| &\lesssim \sum_{a=m_2+1}^{n-m_1-1} \sum_{k: M_k \in S_{j+1,i}^a} \lambda_k j^{n'-m'_1-a+1} i^{n'-a-m'_1} \\ &= \sum_{a=m_2+1}^{n-m_1-1} \nu(S_{j+1,i}^a) j^{n'-m'_1-a+1} i^{n'-a-m'_1} \lesssim \sum_{a=m_2+1}^{n-m_1-1} j^{2(m'_1+a-n')} i^{m'_1+a-n'} j^{n'-m'_1-a+1} i^{n'-a-m'_1} \\ &= \sum_{a=m_2+1}^{n-m_1-1} j^{m'_1-n'+a+1} \lesssim 1 \end{aligned}$$

and

$$\begin{aligned}
& \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} \sum_{k: M_k \in S_{j+1, i+b}^a} \lambda_k \frac{\det(M_k)}{\det(A)} |A^{-1} - M_k^{-1}| \\
& \lesssim \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} \sum_{k: M_k \in S_{j+1, i+b}^a} \lambda_k j^{n'-m'_1-a+1} (i+b)^{n'-a} i^{-m'_1} \\
& = \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} \nu(S_{j+1, i+b}^a) j^{n'-m'_1-a+1} (i+b)^{n'-a} i^{-m'_1} \\
& \lesssim \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} j^{m'_1+2m'_2-n'-a+1} (i+b)^{2m'_2-2a} \lesssim j^{-1} \sum_{b=1}^{j-i+1} (i+b)^{-2} \lesssim 1.
\end{aligned}$$

Therefore, we get

$$\sum_{k=1}^N \lambda_k \frac{\det(M_k)}{\det(A)} |A^{-1} - M_k^{-1}| \lesssim 1.$$

The proof is finished.  $\square$

The proof of the next lemma is analogous to the one of Lemma 16, but using Lemma 13 in the induction instead of Lemma 14.

**Lemma 17.** *Let  $i, j \in \mathbb{N}$ ,  $i \leq j$ , and  $A \in B_i^j$ . Then, there exists a laminate  $\nu = \sum_{k=1}^N \lambda_k \delta_{M_k}$  such that*

- a)  $\bar{\nu} = A$ ,
- b)  $\text{supp}(\nu) \subset A_{j+1}^{j+1} \cup \bigcup_{b=0}^{j-i+1} B_{i+b}^{j+1} \cup \bigcup_{b=0}^{j-i+1} \bigcup_{a=m_2+1}^{n-m_1-1} S_{i+b, j+1}^a$ ,
- c)  $\nu(A_{j+1}^{j+1}) \lesssim j^{m'_1+m'_2-n'}$ ,
- d)  $\nu(B_i^{j+1}) - \left(\frac{j+2}{j+3}\right)^{n'-m'_2} \lesssim (ij)^{-2}$ ,
- e)  $\nu(B_{i+b}^{j+1}) \lesssim ((i+b)(j+3))^{2(m'_1+m'_2-n')}$ , for  $b \in \{1, \dots, j-i+1\}$ ,
- f)  $\nu(S_{i, j+1}^a) \lesssim j^{m'_2-a}$  for  $a \in \{m_2+1, \dots, n-m_1-1\}$ ,
- g)  $\nu(S_{i+b, j+1}^a) \lesssim j^{m'_1+m'_2-n'} ((i+b-1)^2(j+1))^{m'_1+a-n'}$  for  $a \in \{m_2+1, \dots, n-m_1-1\}$  and  $b \in \{1, \dots, j-i+1\}$ ,
- h)  $\sum_{k=1}^N \lambda_k |A - M_k| \lesssim 1$ ,
- i)  $\frac{1}{\det(A)} \sum_{k=1}^N \lambda_k \det(M_k) |A^{-1} - M_k^{-1}| \lesssim j^{-2}$ .

We recall now two results from [14] that will be used to prove Theorem 1.

**Proposition 18.** *Let  $N \in \mathbb{N}$ ,  $A_1, \dots, A_N \in \Gamma_+$  and  $L \geq 1$  be such that*

$$\sigma_1(A_i) \geq L^{-1}, \quad |A_i| \leq L, \quad i = 1, \dots, N.$$

*Consider  $\alpha_1, \dots, \alpha_N \geq 0$  such that  $\nu := \sum_{i=1}^N \alpha_i \delta_{A_i}$  is in  $\mathcal{L}(\mathbb{R}^{n \times n})$  and call  $A := \overline{\nu}$ . Then, for every  $\alpha \in (0, 1)$ ,  $0 < \delta < \frac{1}{2} \min\{L^{-1}, \min_{1 \leq i < j \leq N} |A_i - A_j|\}$  and every bounded open set  $\Omega \subset \mathbb{R}^n$ , there exists a piecewise affine bi-Lipschitz homeomorphism  $f : \Omega \rightarrow A\Omega$  such that*

- (a)  $f = \nabla u$  for some  $u \in W^{2,\infty}(\Omega)$ ,
- (b)  $f(x) = Ax$  for  $x \in \partial\Omega$ ,
- (c)  $\|f - A\|_{C^\alpha(\overline{\Omega})} < \delta$ ,
- (d)  $\|f^{-1} - A^{-1}\|_{C^\alpha(A\overline{\Omega})} < \delta$ ,
- (e)  $|\{x \in \Omega : |Df(x) - A_i| < \delta\}| = \alpha_i |\Omega|$  for all  $i = 1, \dots, N$ .

**Lemma 19.** *Let  $f : \overline{\Omega} \rightarrow \mathbb{R}^n$  be a homeomorphism such that  $f$  and  $f^{-1}$  are  $C^\alpha$  for some  $\alpha \in (0, 1]$ . Let  $\{\omega_i\}_{i \in \mathbb{N}} \subset \Omega$  be a family of pairwise disjoint open sets, and for each  $i \in \mathbb{N}$  let  $g_i : \overline{\omega_i} \rightarrow f(\overline{\omega_i})$  be a homeomorphism such that  $g_i = f$  on  $\partial\omega_i$ ,*

$$\sup_{i \in \mathbb{N}} \|f - g_i\|_{C^\alpha(\overline{\omega_i})} < \infty \quad \text{and} \quad \sup_{i \in \mathbb{N}} \|f^{-1} - g_i^{-1}\|_{C^\alpha(f(\overline{\omega_i}))} < \infty.$$

*Then, the function*

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \overline{\Omega} \setminus \bigcup_{i \in \mathbb{N}} \omega_i, \\ g_i(x) & \text{if } x \in \omega_i \text{ for some } i \in \mathbb{N} \end{cases}$$

*is a homeomorphism between  $\overline{\Omega}$  and  $f(\overline{\Omega})$  such that  $\tilde{f}$  and  $\tilde{f}^{-1}$  are  $C^\alpha$  and*

$$\|f - \tilde{f}\|_{C^\alpha(\overline{\Omega})} \leq 2 \sup_{i \in \mathbb{N}} \|f - g_i\|_{C^\alpha(\overline{\omega_i})}, \quad \|f^{-1} - \tilde{f}^{-1}\|_{C^\alpha(\overline{\Omega})} \leq 2 \sup_{i \in \mathbb{N}} \|f^{-1} - g_i^{-1}\|_{C^\alpha(f(\overline{\omega_i}))}.$$

We construct now two families of uniformly bounded constants that will be used in the proof of Theorem 1.

**Lemma 20.** *Let  $\varepsilon' > 0$  and  $\tilde{C} > 1$ . Define the sequences  $\{C_{j,i}^1\}_{i=1,\dots,j}^{j \in \mathbb{N}}$ ,  $\{C_{j,i}^2\}_{i=0,\dots,j}^{j \in \mathbb{N}}$  and  $\{M_j\}_{j \in \mathbb{N}}$  as follows:*

a)  $C_{1,1}^1 = 4^n$ ,  $C_{1,1}^2 = 4^n$ ,  $C_{j,0}^2 = 0$  for  $j \in \mathbb{N}$ .

*Given  $j \in \mathbb{N}$ , assuming  $C_{j,i}^1$  and  $C_{j,i}^2$  have been defined for all  $i \in \{1, \dots, j\}$ , set*

b)  $M_j = \max_{i=1,\dots,j} C_{j,i}^1$ ,

c)  $C_{j+1,i}^1 = C_{j,i}^1 + j^{-2} \tilde{C} M_j + \tilde{C} C_{j,i-1}^2 (j+2-i)^{-2}$ , for  $i = 1, \dots, j$ ,

$$d) C_{j+1,j+1}^1 = \tilde{C} \left( M_j + C_{j,j}^2 \right) j^{-\varepsilon'},$$

$$e) C_{j+1,i}^2 = C_{j,i-1}^2 \left( 1 + \tilde{C} i^{-2} \right) + \tilde{C} M_j i^{-2+\varepsilon'}, \text{ for } i = 1, \dots, j,$$

$$f) C_{j+1,j+1}^2 = C_{j,j}^2 \left( 1 + \tilde{C} j^{-2} \right) + \tilde{C} M_j j^{-4},$$

Then

$$\sup_{j \in \mathbb{N}} M_j < \infty \text{ and } \sup_{\substack{j \in \mathbb{N} \\ i=0, \dots, j}} C_{j,i}^2 < \infty.$$

*Proof.* Clearly  $M_j \leq M_{j+1}$  for  $j \in \mathbb{N}$ . From *f)* we obtain by induction on  $j$  that

$$C_{j+1,j+1}^2 = C_{1,1}^2 \prod_{k=1}^j (1 + \tilde{C} k^{-2}) + \sum_{\ell=1}^j \left[ \prod_{k=\ell+1}^j (1 + \tilde{C} k^{-2}) \right] \tilde{C} M_\ell \ell^{-4}.$$

Hence

$$(22) \quad C_{j+1,j+1}^2 \lesssim \sum_{\ell=1}^j M_\ell \ell^{-4} \lesssim M_j.$$

Therefore, from *d)* we get

$$(23) \quad C_{j+1,j+1}^1 \lesssim M_j j^{-\varepsilon'}.$$

On the other hand, from *e)* by induction on  $j$ , we have, for  $i = 1, \dots, j$ , that

$$(24) \quad C_{j+1,i}^2 = \sum_{\ell=1}^i \left[ \prod_{k=\ell+1}^i (1 + \tilde{C} k^{-2}) \right] \tilde{C} M_{j+\ell-i} \ell^{-2+\varepsilon'} \lesssim \sum_{\ell=1}^i M_{j+\ell-i} \ell^{-2+\varepsilon'}.$$

By *c)* we obtain by induction on  $j \geq i-1$  that

$$(25) \quad C_{j+1,i}^1 = C_{i,i}^1 + \sum_{\ell=i}^j \tilde{C} (M_\ell \ell^{-2} + (\ell+2-i)^{-2} C_{\ell,i-1}^2).$$

For  $r \geq 1$  we use that

$$-s^2 + (r-2)s + 2r \geq \begin{cases} \frac{r(s+1)}{2} & \text{for } 0 \leq s \leq \frac{r}{2}, \\ \frac{r(r-s)}{2} & \text{for } \frac{r}{2} \leq s \leq r-1 \end{cases}$$

to get

$$(26) \quad \begin{aligned} \sum_{\ell=i}^j \left[ (\ell+2-i)^{-2} \sum_{k=1}^{i-1} M_{\ell+k-i} k^{-2+\varepsilon'} \right] &\leq \sum_{r=1}^{j-1} M_r \sum_{s=0}^{r-1} (s+2)^{-2} (r-s)^{-2+\varepsilon'} \\ &\leq \sum_{r=1}^{j-1} M_r \sum_{s=0}^{r-1} (-s^2 + (r-2)s + 2r)^{-2+\varepsilon'} \lesssim \sum_{\ell=1}^{j-1} M_\ell \ell^{-2+\varepsilon'}. \end{aligned}$$

Now, we use (23), (24), (25) and (26) to get that there exists a constant  $C'$  depending only on  $n$  such that

$$C_{j+1,i}^1 \leq C_{i,i}^1 + C' \sum_{\ell=1}^j M_\ell \ell^{-2+\varepsilon'}.$$

Let  $j_{\varepsilon'}$  be such that for all  $j \geq j_{\varepsilon'}$

$$C_{j+1,j+1}^1 \leq M_j,$$

which is possible thanks to (23). Define the family of constants  $\{C_{j,i}^3\}_{\substack{j \in \mathbb{N} \\ i=1,\dots,j}}$  as follows

$$C_{j,i}^3 = \begin{cases} C_{j,i}^1 & \text{if } j \leq j_{\varepsilon'} \text{ or } i = j, \\ C_{i,i}^1 + C' \sum_{\ell=1}^{j-1} M_\ell \ell^{-2+\varepsilon'} & \text{if } j > \max\{i, j_{\varepsilon'}\}. \end{cases}$$

Hence,  $C_{j,i}^1 \leq C_{j,i}^3$  for all  $j \in \mathbb{N}$  and  $i = 1, \dots, j$ . Define  $M'_j = \max_{i=1,\dots,j} C_{j,i}^3$  and let  $i_j \in \{1, \dots, j\}$  be such that  $M'_j = C_{j,i_j}^3$ . First, we note that  $M'_j < M'_{j+1}$  for all  $j > j_{\varepsilon'}$ . Fix  $j > j_{\varepsilon'}$ ; as

$$C_{j+1,j+1}^3 = C_{j+1,j+1}^1 \leq M_j \leq M'_j \text{ and } M'_j < M'_{j+1},$$

it is clear that  $i_{j+1} \leq j$ . We also see that

$$C_{j+1,i_j}^3 = C_{j,i_j}^3 + C' M_j j^{-2+\varepsilon'} = M'_j + C' M_j j^{-2+\varepsilon'},$$

and for all  $i \leq j$  we have

$$C_{j+1,i}^3 = C_{j,i}^3 + C' M_j j^{-2+\varepsilon'}.$$

Therefore  $M'_{j+1} = C_{j+1,i_j}^3$ , so we can take  $i_{j+1} = i_j$ . By induction, we can take  $i_j = i_{j_{\varepsilon'}+1}$  for all  $j > j_{\varepsilon'}$ , and  $M'_j = C_{j,i_{j_{\varepsilon'}+1}}^3$ . Hence, there exists  $C > 0$  such that

$$M'_{j+1} \leq C \sum_{\ell=1}^j M_\ell \ell^{-2+\varepsilon'} \leq C \sum_{\ell=1}^j M'_\ell \ell^{-2+\varepsilon'}.$$

Define for  $j \leq j_{\varepsilon'} + 1$ ,  $\tilde{M}_j = M'_j$  and for  $j \geq j_{\varepsilon'} + 1$

$$\tilde{M}_{j+1} = C \sum_{\ell=1}^j \tilde{M}_\ell \ell^{-2+\varepsilon'},$$

so  $M'_j \leq \tilde{M}_j$  for all  $j \in \mathbb{N}$ . Now, we observe that for  $j \geq j_{\varepsilon'} + 2$  we have

$$\tilde{M}_{j+1} = \tilde{M}_j (1 + C j^{-2+\varepsilon'}).$$

Therefore

$$\sup_{j \in \mathbb{N}} M_j \leq \sup_{j \in \mathbb{N}} M'_j \leq \sup_{j \in \mathbb{N}} \tilde{M}_j < \infty.$$



Using (22), (24) and that  $\sup_{j \in \mathbb{N}} M_j < \infty$  we have

$$\sup_{\substack{j \in \mathbb{N} \\ i=1, \dots, j}} C_{j,i}^2 < \infty.$$

Here concludes the proof.  $\square$

Finally, we combine all the previous results to prove Theorem 1.

*Proof of Theorem 1.* In this proof, expressions like  $\{x \in \Omega : f(x) \in A\}$  will be abbreviated as  $\{f(x) \in A\}$ . Given  $\varepsilon' > 0$  small enough to have  $\sum_{k=1}^{\infty} k^{-2+\varepsilon'} < 2$ , we will construct a sequence  $\{f_j\}_{j \in \mathbb{N}} \subset W^{1,1}(\Omega, \Omega)$  of piecewise affine Lipschitz homeomorphisms such that  $f_0 = \text{id}$  and, when we take  $\{C_{j,i}^1\}_{\substack{j \in \mathbb{N} \\ i=1, \dots, j}}$ ,  $\{C_{j,i}^2\}_{\substack{j \in \mathbb{N} \\ i=0, \dots, j}}$  the families of constants in Lemma 20 and we denote  $\Omega_S^j = \{Df_j(x) \in S\}$  for each  $S \subset \Gamma_+$ , we have

- i)  $f_j = \text{id}$  on  $\partial\Omega$ ,
- ii)  $\|f_j - f_{j-1}\|_{C^\alpha(\overline{\Omega})} < 2^{-j}\varepsilon$  and  $\|f_j^{-1} - f_{j-1}^{-1}\|_{C^\alpha(\overline{\Omega})} < 2^{-j}\varepsilon$ ,
- iii)  $Df_j(x) \in \bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} (A_j^i \cup B_i^j \cup S_{j,i}^a \cup S_{i,j}^a)$ ,
- iv)  $\int_{\Omega} |Df_j(x) - Df_{j-1}(x)| dx \lesssim j^{-2}|\Omega|$ ,
- v)  $\int_{\Omega} |Df_j^{-1}(y) - Df_{j-1}^{-1}(y)| dy \lesssim j^{-2}|\Omega|$ ,
- vi)  $\frac{|\Omega_{A_j^i}^j|}{|\Omega|} \leq C_{j,i}^1 i^{-m'_1-2+\varepsilon'}$  for  $i = 1, \dots, j$ ,
- vii)  $\frac{|\Omega_{B_i^j}^j|}{|\Omega|} \leq C_{j,i}^1 i^{-2+\varepsilon'} (j+2)^{m'_2-n'}$  for  $i = 1, \dots, j$ ,
- viii)  $\frac{|\Omega_{S_{j,i}^a}^j|}{|\Omega|} \leq C_{j,i}^2 (i+2)^{a-n'} (j+1-i)^{-2}$  for  $i = 1, \dots, j$  and  $a \in \{m_2+1, \dots, n-m_1-1\}$ ,
- ix)  $\frac{|\Omega_{S_{i,j}^a}^j|}{|\Omega|} \leq C_{j,i}^2 (j+2)^{a-n'} (j+1-i)^{-2}$  for  $i = 1, \dots, j$  and  $a \in \{m_2+1, \dots, n-m_1-1\}$ .

One constructed such sequence  $\{f_j\}$ , we have that it converges in the  $C^\alpha$  and in the  $W^{1,1}$  norm to a bi-Sobolev homeomorphism  $f : \Omega \rightarrow \Omega$ ; see, if necessary the proof of [14, Th. 1] for the details of the limit passage. Moreover, it is immediate from i), ii), that  $f$  satisfies i) and ii) of Theorem 1. Recall that the bounds of  $S_{i,k}^a$  only have sense if  $m_1 + m_2 \leq n - 1$ ; otherwise, these sets are empty. From iii), vii), viii) and ix) we obtain

$$(27) \quad 1 - \frac{|\Omega_{\bigcup_{i=1}^j A_j^i}^j|}{|\Omega|} \lesssim \sum_{i=1}^j \left[ i^{-2+\varepsilon'} j^{m'_2-n'} + (j+1-i)^{-2} \left( \sum_{a=m_2+1}^{n-m_1-1} i^{a-n'} + j^{a-n'} \right) \right] \lesssim j^{-1}.$$

For a subsequence,  $Df_j \rightarrow Df$  a.e., so, thanks to the continuity of the singular values and using that  $\Gamma_+$  is closed we obtain that  $Df \in \Gamma_+$  and, thanks to (27), we also get that  $\text{rank}(Df) = m_1$  a.e. in  $\Omega$ . On the other hand,

$$\begin{aligned}
& 1 - \frac{\left| \left\{ Df_j^{-1}(y) \in \bigcup_{i=1}^j (B_i^j)^{-1} \right\} \right|}{|\Omega|} \\
&= \frac{\left| \left\{ Df_j^{-1}(y) \in \bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} (A_j^i \cup S_{i,j}^a \cup S_{j,i}^a)^{-1} \right\} \right|}{|\Omega|} \\
&= \frac{1}{|\Omega|} \left| f_j \left( \left\{ Df_j(x) \in \bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} A_j^i \cup S_{i,j}^a \cup S_{j,i}^a \right\} \right) \right| \\
&= \frac{1}{|\Omega|} \int_{\left\{ Df_j(x) \in \bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} A_j^i \cup S_{i,j}^a \cup S_{j,i}^a \right\}} \det(Df_j(x)) dx.
\end{aligned}$$

Now, we split the integral over the different sets and we use the control that we have over the determinant in the different sets (the second part with  $M \in B_i^j$  will be used later):

$$(28) \quad \det(M) \lesssim \begin{cases} j^{m'_1-n'} i^{m'_1} & \text{if } M \in A_j^i, \\ j^{n'-m'_2} i^{-m'_2} & \text{if } M \in B_i^j, \\ k^{-a} i^{n'-a} & \text{if } M \in S_{k,i}^a. \end{cases}$$

Therefore using vi), viii) and ix) we get

$$\begin{aligned}
& 1 - \frac{\left| \left\{ Df_j^{-1}(x) \in \bigcup_{i=1}^j (B_i^j)^{-1} \right\} \right|}{|\Omega|} \\
&\lesssim \sum_{i=1}^j \left[ i^{-m'_1-2+\varepsilon'} j^{m'_1-n'} i^{m'_1} + (j+1-i)^{-2} \sum_{a=m_2+1}^{n-m_1-1} \left( i^{a-n'} j^{-a} i^{n'-a} + j^{a-n'} i^{-a} j^{n'-a} \right) \right] \lesssim j^{-1}.
\end{aligned}$$

The same argument as before shows that  $\text{rank}(Df^{-1}(y)) = m_2$  a.e.  $y \in \Omega$ . Hence, iii) of Theorem 1 is proved.

Let  $\varepsilon, t > 0$  and pick  $j > t$ ; then using that

$$|M| \leq \begin{cases} i+2 & \text{if } M \in A_j^i, \\ j+2 & \text{if } M \in B_i^j, \\ i+2 & \text{if } M \in S_{k,i}^a, \end{cases}$$

we have

$$\begin{aligned}
\frac{|\{|Df_j(x)| > t\}|}{|\Omega|} &\leq \sum_{i=\lfloor t \rfloor - 1}^j \left[ \frac{|\Omega_{A_j^i}^j|}{|\Omega|} + \sum_{a=m_2+1}^{n-m_1-1} \frac{|\Omega_{S_{j,i}^a}^j|}{|\Omega|} \right] + \sum_{i=1}^j \left[ \frac{|\Omega_{B_i^j}^j|}{|\Omega|} + \sum_{a=m_2+1}^{n-m_1-1} \frac{|\Omega_{S_{i,j}^a}^j|}{|\Omega|} \right] \\
&\lesssim \sum_{i=\lfloor t \rfloor - 1}^j \left[ i^{-m'_1-2+\varepsilon'} + \sum_{a=m_2+1}^{n-m_1-1} i^{a-n'} (j+1-i)^{-2} \right] \\
&\quad + \sum_{i=1}^j \left[ i^{-2+\varepsilon'} j^{m'_2-n'} + \sum_{a=m_2+1}^{n-m_1-1} j^{a-n'} (j+1-i)^{-2} \right].
\end{aligned}$$

Hence, using  $\sum_{i=\lfloor t \rfloor - 1}^j i^{-m'_1-1} (j+1-i)^{-2} \lesssim t^{-m'_1-1} \leq t^{-m'_1-1+\varepsilon'}$  and  $m'_1 + m'_2 \leq n' - 1$  we get

$$\frac{|\{|Df_j(x)| > t\}|}{|\Omega|} \lesssim t^{-m'_1-1+\varepsilon'},$$

and, hence, since we have proved the bound for all  $\varepsilon' > 0$  we have  $f \in W^{1,p}(\Omega, \mathbb{R}^n)$  for all  $p < m'_1 + 1$ .

Next, for the inverse we will use the bounds

$$|M^{-1}| \leq \begin{cases} j+2 & \text{if } M \in A_j^i, \\ i+2 & \text{if } M \in B_i^j, \\ k+2 & \text{if } M \in S_{k,i}^a. \end{cases}$$

Therefore we get

$$\begin{aligned}
\frac{|\{|Df_j^{-1}(y)| > t\}|}{|\Omega|} &= \frac{|f_j(\{|Df_j(x)|^{-1} > t\})|}{|\Omega|} \\
&\leq \sum_{i=1}^j \left[ \frac{|f_j(\Omega_{A_j^i}^j)|}{|\Omega|} + \sum_{a=m_2+1}^{n-m_1-1} \frac{|f_j(\Omega_{S_{j,i}^a}^j)|}{|\Omega|} \right] + \sum_{i=\lfloor t \rfloor - 1}^j \left[ \frac{|f_j(\Omega_{B_i^j}^j)|}{|\Omega|} + \sum_{a=m_2+1}^{n-m_1-1} \frac{|f_j(\Omega_{S_{i,j}^a}^j)|}{|\Omega|} \right].
\end{aligned}$$

Therefore, using (28) we get

$$\begin{aligned}
\frac{|\{|Df_j^{-1}(y)| > t\}|}{|\Omega|} &\lesssim \sum_{i=1}^j \left[ j^{m'_1-n'} i^{m'_1} i^{-m'_1-2+\varepsilon'} + \sum_{a=m_2+1}^{n-m_1-1} j^{-a} i^{n'-a} i^{a-n'} (j+1-i)^{-2} \right] \\
&\quad + \sum_{i=\lfloor t \rfloor - 1}^j \left[ j^{n'-m'_2} i^{-m'_2} i^{-2+\varepsilon'} j^{m'_2-n'} + \sum_{a=m_2+1}^{n-m_1-1} i^{-a} j^{n'-a} j^{a-n'} (j+1-i)^{-2} \right] \lesssim t^{-m'_2-1+\varepsilon'}.
\end{aligned}$$

So, we have  $f^{-1} \in W^{1,q}(\Omega, \mathbb{R}^n)$  for all  $q < m_2 + 1$ , and therefore, part *iv*) of Theorem 1 is proved.

Hence, to prove the theorem it is enough to construct the sequence  $\{f_j\}$ . Let  $f_0 = \text{id}$  and proceeding as in Lemma 13 we construct a laminate  $\nu_1$  such that  $\overline{\nu}_1 = I$  and

$$\text{supp}(\nu_1) \subset A_1^1 \cup B_1^1 \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{1,1}^a.$$

Now apply Proposition 18 with  $\delta$  small enough to have  $Df_1(x) \in A_1^1 \cup B_1^1 \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{1,1}^a$  for almost every  $x \in \Omega$ ; this is possible because Proposition 18 gives us that  $Df(x) \in \Gamma_+$  a.e.  $x \in \Omega$ ; and, for  $i, k \in \mathbb{N}$ ,  $a \in \{m_2+1, \dots, n-m_1-1\}$ , the sets  $A_i^k, B_i^k, S_{i,k}^a$  are open in  $\Gamma_+$ . We do now the inductive step: suppose that we have constructed  $f_j$ . As  $f_j$  is piecewise affine, there exist families  $\{A_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^{n \times n}$ ,  $\{b_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$  and  $\{\omega_k\}_{k \in \mathbb{N}}$  of disjoint sets such that  $|\Omega \setminus (\bigcup_{k \in \mathbb{N}} \omega_k)| = 0$  and

$$f_j(x) = A_k x + b_k \text{ in } \omega_k.$$

For each  $k$ , let  $\nu_k$  the laminate given by Lemma 15 if  $A_k \in \bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a \cup S_{i,j}^a$ , the one given by Lemma 16 if  $A_k \in \bigcup_{i=1}^j A_i^j$  and the one given by Lemma 17 if  $A_k \in \bigcup_{i=1}^j B_i^j$ . For each  $k$ , let  $g_k$  be the homeomorphism given by Proposition 18 corresponding to  $\nu_k$  that is equal to  $A_k x + b_k$  on the border of  $\omega_k$  with  $\delta_k > 0$  being as small as we need in the rest of the proof. First let  $\delta_k$  be small enough to get

$$Dg_k(x) \in \bigcup_{i=1}^{j+1} \bigcup_{a=m_2+1}^{n-m_1-1} \left( A_{j+1}^i \cup B_i^{j+1} \cup S_{j+1,i}^a \cup S_{i,j+1}^a \right) \text{ for almost every } x \in \omega_k,$$

and such that for all  $k \in \mathbb{N}$  we have  $\delta_k < 2^{-j-1}\varepsilon$ . Then, we define

$$f_{j+1} = \begin{cases} g_k & \text{in } \omega_k \text{ for some } k \in \mathbb{N}, \\ f_j & \text{in } \Omega \setminus \bigcup_{k \in \mathbb{N}} \omega_k. \end{cases}$$

It is obvious that  $f_{j+1}$  satisfy i) and iii); using Lemma 19 we see that it is a homeomorphism, and by Proposition 18 and Lemma 19 we have ii). Now we prove iv) and v). Choose  $\delta_k$  such that, if  $\nu_k = \sum_{\ell=1}^{N_k} \lambda_{k,\ell} \delta_{M_{k,\ell}}$ ,

$$\delta_k < \min_{\ell \in \{1, \dots, N_k\}} |A_k - M_{k,\ell}|,$$

and for  $\ell = 1, \dots, N_k$ , using part (e) of Proposition 18 we have that it also holds

$$|Dg_k^{-1}(y) - M_{k,\ell}^{-1}| < |A_k^{-1}y - M_{k,\ell}^{-1}| \text{ in } g_k(\{x \in \omega_k : |Dg_k(x) - M_{k,\ell}| < \delta_k\}).$$

Denote by  $\omega_{k,\ell}$  the set  $\{x \in \omega_k : |Dg_k(x) - M_{k,\ell}| < \delta_k\}$ . Recall, from Proposition 18 (e), that  $|\omega_{k,\ell}| = \lambda_{k,\ell} |\omega_k|$ . Then, using parts h) and i) of Lemma 16, we have that for those  $k \in \mathbb{N}$  such that  $A_k \in \bigcup_{i=1}^j A_i^j$ ,

$$\begin{aligned} \int_{\omega_k} |A_k - Dg_k(x)| dx &\leq \sum_{\ell=1}^{N_k} \int_{\omega_{k,\ell}} (|A_k - M_{k,\ell}| + |M_{k,\ell} - Dg_k(x)|) dx \\ &\lesssim \sum_{\ell=1}^{N_k} \int_{\omega_{k,\ell}} |A_k - M_{k,\ell}| dx = \sum_{\ell=1}^{N_k} \lambda_{k,\ell} |A_k - M_{k,\ell}| |\omega_k| \lesssim j^{-2} |\omega_k|, \end{aligned}$$

and, also using  $|g_k(\omega_{k,\ell})| \lesssim \det(M_{k,\ell})|\omega_{k,\ell}| = \det(M_{k,\ell})\lambda_{k,\ell}|\omega_k|$  we get

$$\begin{aligned} \int_{g_k(\omega_k)} |A_k^{-1} - Dg_k^{-1}(y)| dx &\leq \sum_{\ell=1}^{N_k} \int_{g_k(\omega_{k,\ell})} \left( |A_k^{-1} - M_{k,\ell}^{-1}| + |M_{k,\ell}^{-1} - Dg_k^{-1}(y)| \right) dy \\ &\lesssim \sum_{\ell=1}^{N_k} \int_{g_k(\omega_{k,\ell})} |A_k^{-1} - M_{k,\ell}^{-1}| dx \lesssim \sum_{\ell=1}^{N_k} \lambda_{k,\ell} \det(M_{k,\ell}) |\omega_k| |A_k^{-1} - M_{k,\ell}^{-1}| \lesssim \det(A_k) |\omega_k| = |g_k(\omega_k)|. \end{aligned}$$

Proceeding in the same way as before we obtain that for those  $k \in \mathbb{N}$  such that  $A_k \in \bigcup_{i=1}^j B_i^j$  we have

$$\int_{\omega_k} |A_k - Dg_k(x)| dx \lesssim |\omega_k|$$

and

$$\int_{g_k(\omega_k)} |A_k^{-1} - Dg_k^{-1}(y)| dy \lesssim j^{-2} |g_k(\omega_k)|.$$

Similarly, for  $k \in \mathbb{N}$  such that  $A_k \in \bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a \cup S_{i,j}^a$  we have

$$\int_{\omega_k} |A_k - Dg_k(x)| dx \lesssim |\omega_k|$$

and

$$\int_{g_k(\omega_k)} |A_k^{-1} - Dg_k^{-1}(y)| dy \lesssim |g_k(\omega_k)|.$$

Now we combine the last equations, and we use that  $Df_{j+1}(x) = Dg_k(x)$  and  $Df_j(x) = A_k$  for  $x$  in  $\omega_k$ , to prove iv) and v):

$$\begin{aligned} \int_{\Omega} |Df_{j+1} - Df_j| dx &= \sum_{k \in \mathbb{N}} \int_{\omega_k} |Dg_k - A_k| dx \\ &\lesssim \sum_{\substack{k \in \mathbb{N} \\ A_k \in \bigcup_{i=1}^j A_j^i}} j^{-2} |\omega_k| + \sum_{\substack{k \in \mathbb{N} \\ A_k \in \bigcup_{i=1}^j B_i^j}} |\omega_k| + \sum_{\substack{k \in \mathbb{N} \\ A_k \in \bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a \cup S_{i,j}^a}} |\omega_k| \\ &= j^{-2} |\Omega_{\bigcup_{i=1}^j A_j^i}^j| + |\Omega_{\bigcup_{i=1}^j B_i^j}^j| + |\Omega_{\bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a \cup S_{i,j}^a}^j|. \end{aligned}$$

Now, we use vi)–ix), that  $m'_2 - n' \leq -2$  and we recall that in the bounds for the sets  $S_{k,i}^a$  we can suppose  $n' = n$ ,  $m'_1 = m_1$  and  $m'_2 = m_2$ , because otherwise they are empty. Therefore, we obtain

$$\begin{aligned} &\frac{1}{|\Omega|} \int_{\Omega} |Df_{j+1} - Df_j| dx \\ &\lesssim \sum_{i=1}^j \left( j^{-2} i^{-m'_1-2+\varepsilon'} + i^{-2+\varepsilon'} j^{m'_2-n'} + \sum_{a=m_2+1}^{n-m_1-1} \left[ i^{a-n'} (j+1-i)^{-2} + j^{a-n'} (j+1-i)^{-2} \right] \right) \\ &\lesssim j^{-2} + \sum_{i=1}^j i^{-2} (j+1-i)^{-2} \lesssim j^{-2}. \end{aligned}$$

So iv) is proved. On the other hand, using  $f_{j+1}(\omega_k) = g_k(\omega_k) = f_j(\omega_k)$  we have

$$\begin{aligned}
\int_{\Omega} |Df_{j+1}^{-1} - Df_j^{-1}| dy &= \sum_{k \in \mathbb{N}} \int_{g_k(\omega_k)} |Dg_k^{-1} - A_k^{-1}| dy \\
&\lesssim \sum_{\substack{k \in \mathbb{N} \\ A_k \in \bigcup_{i=1}^j A_j^i}} |g_k(\omega_k)| + \sum_{\substack{k \in \mathbb{N} \\ A_k \in \bigcup_{i=1}^j B_i^j}} j^{-2} |g_k(\omega_k)| + \sum_{\substack{k \in \mathbb{N} \\ A_k \in \bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a \cup S_{i,j}^a}} |g_k(\omega_k)| \\
&= \left| f_j \left( \Omega_{\bigcup_{i=1}^j A_j^i} \right) \right| + j^{-2} \left| f_j \left( \Omega_{\bigcup_{i=1}^j B_i^j} \right) \right| + \left| f_j \left( \Omega_{\bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a \cup S_{i,j}^a} \right) \right|.
\end{aligned}$$

Using again vi)–ix) and that we have a control over the determinant of  $Df_j(x)$  when  $x$  is in the different sets, see (28), we obtain

$$\begin{aligned}
\frac{1}{|\Omega|} \int_{\Omega} |Df_{j+1}^{-1} - Df_j^{-1}| dy &\lesssim \sum_{i=1}^j \left( j^{m'_1 - n'} i^{m'_1} i^{-m'_1 - 2 + \varepsilon'} + j^{-2} j^{n' - m'_2} i^{-m'_2} i^{-2 + \varepsilon'} j^{m'_2 - n'} \right. \\
&\quad \left. + \sum_{a=m_2+1}^{n-m_1-1} \left[ j^{-a} i^{n'-a} i^{a-n'} (j+1-i)^{-2} + i^{-a} j^{n'-a} j^{a-n'} (j+1-i)^{-2} \right] \right) \\
&\lesssim \left( j^{-2} + \sum_{i=1}^j i^{-2} (j+1-i)^{-2} \right) \lesssim j^{-2}.
\end{aligned}$$

Hence, we have proved v). Finally, we suppose that vi)–ix) holds for  $j$  and we prove them for  $j+1$ . Let  $C$  be a constant bigger than the ones appearing in Lemmas 15, 16 and 17.

Let  $i = 1, \dots, j$ ; from the construction of  $\nu_k$  we can see that if  $\text{supp}(\nu_k) \cap A_{j+1}^i \neq \emptyset$  then

$$\bar{\nu}_k = A_k \in \bigcup_{l=1}^i A_j^l \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i-1}^a.$$

On the other hand, using part (e) of Proposition 18 we get

$$\begin{aligned}
|\Omega_{A_{j+1}^i}^{j+1}| &= \sum_{k: A_k \in \bigcup_{l=1}^i A_j^l \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i-1}^a} \sum_{\ell: M_{k,\ell} \in A_{j+1}^i} \lambda_{k,\ell} |\omega_k| \\
&= \sum_{k: A_k \in \bigcup_{l=1}^i A_j^l \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i-1}^a} \nu_k(A_{j+1}^i) |\omega_k| \\
&= \sum_{l=1}^i \sum_{k: A_k \in A_j^l} \nu_k(A_{j+1}^i) |\omega_k| + \sum_{a=m_2+1}^{n-m_1-1} \sum_{k: A_k \in S_{j,i-1}^a} \nu_k(A_{j+1}^i) |\omega_k|.
\end{aligned}$$

Now, we use the control that we have over  $\nu_k(A_{j+1}^i)$  given by Lemmas 15 and 16, and also

that given  $S \subset \Gamma_+$  we have  $\sum_{k:A_k \in S} |\omega_k| = |\Omega_S^j|$ . Therefore, we obtain

$$\begin{aligned} |\Omega_{A_{j+1}^i}^{j+1}| &\leq \sum_{l=1}^{i-1} \sum_{k:A_k \in A_j^l} C j^{2(m'_1+m'_2-n')} \frac{l^{m'_1}}{i^{m'_1+2}} |\omega_k| + \sum_{k:A_k \in A_j^i} |\omega_k| + \sum_{a=m_2+1}^{n-m_1-1} \sum_{k:A_k \in S_{j,i-1}^a} C i^{a+m'_1-n'} |\omega_k| \\ &= \sum_{l=1}^{i-1} C j^{2(m'_1+m'_2-n')} \frac{l^{m'_1}}{i^{m'_1+2}} |\Omega_{A_j^l}^j| + |\Omega_{A_j^i}^j| + \sum_{a=m_2+1}^{n-m_1-1} C i^{a+m'_1-n'} |\Omega_{S_{j,i-1}^a}^j|. \end{aligned}$$

Let  $M_j$  be as in Lemma 20. By induction we get

$$\begin{aligned} \frac{|\Omega_{A_{j+1}^i}^{j+1}|}{|\Omega|} &\leq \sum_{l=1}^{i-1} C_{j,l}^1 C j^{2(m'_1+m'_2-n')} \frac{l^{m'_1}}{i^{m'_1+2}} l^{-m'_1-2+\varepsilon'} + C_{j,i}^1 i^{-m'_1-2+\varepsilon'} \\ &\quad + \sum_{a=m_2+1}^{n-m_1-1} C_{j,i-1}^2 C i^{a+m'_1-n'} (i+2)^{a-n'} (j+2-i)^{-2} \\ &\leq (C_{j,i}^1 + j^{-2} 2CM_j + nCC_{j,i-1}^2 (j+2-i)^{-2}) i^{-m'_1-2+\varepsilon'} \leq C_{j+1,i}^1 i^{-m'_1-2+\varepsilon'}. \end{aligned}$$

The last estimate comes from c) of Lemma 20; we will use this lemma several times in the rest of the proof with  $\tilde{C} = 16^n C$ . Now, for each  $i = 1, \dots, j$ , and  $k \in \mathbb{N}$  such that  $\text{supp}(\nu_k) \cap B_i^{j+1} \neq \emptyset$ , we have

$$\bar{\nu}_k = A_k \in \bigcup_{l=1}^i B_l^j \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{i-1,j}^a,$$

hence, proceeding as before we get

$$\begin{aligned} \frac{|\Omega_{B_{j+1}^i}^{j+1}|}{|\Omega|} &\leq \sum_{l=1}^{i-1} C_{j,l}^1 C (i(j+3))^{2(m'_1+m'_2-n')} l^{-2+\varepsilon'} (j+2)^{m'_2-n'} \\ &\quad + C_{j,i}^1 \left( \left( \frac{j+2}{j+3} \right)^{n'-m'_2} + C(ij)^{-2} \right) i^{-2+\varepsilon'} (j+2)^{m'_2-n'} \\ &\quad + \sum_{a=m_2+1}^{n-m_1-1} C_{j,i-1}^2 C (i^2 j)^{m'_2-a} (j+2)^{a-n'} (j+2-i)^{-2} \\ &\leq (C_{j,i}^1 + j^{-2} 2^{n+1} CM_j + n2^{2n} CC_{j,i-1}^2 (j+2-i)^{-2}) i^{-2+\varepsilon'} (j+3)^{m'_2-n'} \\ &\leq C_{j+1,i}^1 i^{-2+\varepsilon'} (j+3)^{m'_2-n'}. \end{aligned}$$

Now, we use that those  $k \in \mathbb{N}$  such that  $\nu_k$  satisfy

$$\text{supp}(\nu_k) \cap A_{j+1}^{j+1} \neq \emptyset \text{ or } \text{supp}(\nu_k) \cap B_{j+1}^{j+1} \neq \emptyset$$

are those that satisfy

$$\bar{\nu}_k = A_k \in \bigcup_{l=1}^j (A_j^l \cup B_l^j) \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,j}^a.$$

Therefore, we obtain

$$\begin{aligned} \frac{|\Omega_{A_{j+1}^{j+1}}^{j+1}|}{|\Omega|} &\leq \sum_{l=1}^j C_{j,l}^1 C \left( j^{2(m'_1+m'_2-n')} \frac{l^{m'_1}}{j^{m'_1+2}} l^{-m'_1-2+\varepsilon'} + j^{m'_1+m'_2-n'} l^{-2+\varepsilon'} (j+2)^{m'_2-n'} \right) \\ &\quad + \sum_{a=m_2+1}^{n-m_1-1} C_{j,j}^2 C j^{a+m'_1-n'} (j+2)^{a-n'} \leq M_j 4C j^{-m'_1-2} + n C C_{j,j}^2 j^{-m'_1-2} \\ &= (4CM_j + n C C_{j,j}^2) j^{-m'_1-2} \leq C_{j+1,j+1}^1 (j+1)^{-m'_1-2+\varepsilon'} \end{aligned}$$

and

$$\begin{aligned} \frac{|\Omega_{B_{j+1}^{j+1}}^{j+1}|}{|\Omega|} &\leq \sum_{l=1}^j C_{j,l}^1 C \left( j^{2m'_1+3m'_2-3n'} l^{m'_1} l^{-m'_1-2+\varepsilon'} + (j(j+1))^{2(m'_1+m'_2-n')} l^{-2+\varepsilon} (j+2)^{m'_2-n'} \right) \\ &\quad + \sum_{a=m_2+1}^{n-m_1-1} C_{j,j}^2 C j^{3(m'_2-a)} (j+2)^{a-n'} \leq M_j 4C j^{m'_2-n'-2} + n C C_{j,j}^2 j^{m'_2-n'-2} \\ &= (4CM_j + n C C_{j,j}^2) j^{m'_2-n'-2} \leq C_{j+1,j+1}^1 (j+1)^{-2+\varepsilon'} (j+3)^{m'_2-n'}. \end{aligned}$$

Hence, vi) and vii) are proved for  $j+1$ .

Finally, proceeding as before, we only have to prove viii) and ix) for  $j+1$ . For each  $i = 1, \dots, j$ ,  $a = m_2 + 1, \dots, n - m_1 - 1$  and  $k \in \mathbb{N}$  such that  $\text{supp}(\nu_k) \cap S_{j+1,i}^a \neq \emptyset$ , we have

$$\bar{\nu}_k = A_k \in \bigcup_{l=1}^i A_j^l \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i-1}^a,$$



hence, using  $2m_2 - a - n \leq a - n - 2$  for  $a \geq m_2 + 1$  we get

$$\begin{aligned}
\frac{|\Omega_{S_{j+1,i}^a}^{j+1}|}{|\Omega|} &\leq \sum_{l=1}^{i-1} C_{j,l}^1 C j^{2(m'_1+m'_2-n')} l^{m'_1} i^{2m'_2-a-n'} l^{-m'_1-2+\varepsilon'} + C_{j,i}^1 C (j^2 i)^{m'_1+a-n'} i^{-m'_1-2+\varepsilon'} \\
&\quad + \sum_{b=m_2+1}^{a-1} C_{j,i-1}^2 C (i-1)^{b-a} (i+1)^{b-n'} (j+2-i)^{-2} \\
&\quad + C_{j,i-1}^2 \left( \left( \frac{i+1}{i+2} \right)^{n'-a} + C(j(i-1))^{-2} \right) (i+1)^{a-n'} (j+2-i)^{-2} \\
&\quad + \sum_{b=a+1}^{n-m_1-1} C_{j,i-1}^2 C (j^2(i-1))^{a-b} (i+1)^{b-n'} (j+2-i)^{-2} \\
&\leq \left( C_{j,i-1}^2 + n4^n C C_{j,i-1}^2 i^{-2} + 4^n C M_j i^{-2+\varepsilon'} \right) (i+2)^{a-n'} (j+2-i)^{-2} \\
&\leq C_{j+1,i}^2 (i+2)^{a-n'} (j+2-i)^{-2}.
\end{aligned}$$

If  $\text{supp}(\nu_k) \cap S_{i,j+1}^a \neq \emptyset$ , then

$$\bar{\nu}_k = A_k \in \bigcup_{l=1}^i B_l^j \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{i-1,j}^a,$$

so, using again  $2m_2 - a - n \leq a - n - 2$  for  $a \geq m_2 + 1$  and that  $a \leq n - m_1 - 1$  we obtain

$$\begin{aligned}
\frac{|\Omega_{S_{i,j+1}^a}^{j+1}|}{|\Omega|} &\leq \sum_{l=1}^{i-1} C_{j,l}^1 C j^{m'_1+m'_2-n'} ((i-1)^2(j+1))^{m'_1+a-n'} l^{-2+\varepsilon'} (j+2)^{m'_2-n'} \\
&\quad + C_{j,i}^1 C j^{m'_2-a} i^{-2+\varepsilon'} (j+2)^{m'_2-n'} + \sum_{b=m_2+1}^{a-1} C_{j,i-1}^2 C j^{b-a} (j+2)^{b-n'} (j+2-i)^{-2} \\
&\quad + C_{j,i-1}^2 \left( \left( \frac{j+2}{j+3} \right)^{n'-a} + C((i-1)j)^{-2} \right) (j+2)^{a-n'} (j+2-i)^{-2} \\
&\quad + \sum_{b=a+1}^{n-m_1-1} C_{j,i-1}^2 C ((i-1)^2 j)^{a-b} (j+2)^{b-n'} (j+2-i)^{-2} \\
&\leq \left( C_{j,i-1}^2 + n8^n C C_{j,i-1}^2 i^{-2} + 8^n C M_j i^{-2+\varepsilon'} \right) (j+3)^{a-n'} (j+2-i)^{-2} \\
&\leq C_{j+1,i}^2 (j+3)^{a-n'} (j+2-i)^{-2}.
\end{aligned}$$

It only remains to estimate  $|\Omega_{S_{j+1,j+1}^a}^{j+1}|$  for  $a \in \{m_2+1, \dots, n-m_1-1\}$ . If  $\text{supp}(\nu_k) \cap S_{j+1,j+1}^a \neq \emptyset$ , then

$$\bar{\nu}_k = A_k \in \bigcup_{l=1}^j \left( A_j^l \cup B_l^j \right) \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,j}^a.$$

Therefore, using  $2m_2 - a - n \leq a - n - 2$  for  $a \geq m_2 + 1$  we have

$$\begin{aligned}
\frac{|\Omega_{S_{j+1,j+1}^a}^{j+1}|}{|\Omega|} &\leq \sum_{l=1}^j C_{j,l}^1 C \left( j^{2(m'_1+m'_2-n')} l^{m'_1} (j+1)^{2m'_2-a-n'} l^{-m'_1-2+\varepsilon'} \right. \\
&\quad \left. + j^{m'_1+m'_2-n'} (j^2(j+1))^{m'_1+a-n'} l^{-2+\varepsilon'} (j+2)^{m'_2-n'} \right) + \sum_{b=m_2+1}^{a-1} C_{j,j}^2 C j^{b-a} (j+2)^{b-n'} \\
&\quad + C_{j,j}^2 \left( \left( \frac{j+2}{j+3} \right)^{n'-a} + C(j^2)^{-2} \right) (j+2)^{a-n'} + \sum_{b=a+1}^{n-m_1-1} C_{j,j}^2 C (j^3)^{a-b} (j+2)^{b-n'} \\
&\leq (C_{j,j}^2 + n8^n C C_{j,j}^2 j^{-2} + 8^n C M_j j^{-4}) (j+3)^{a-n'} \leq C_{j+1,j+1}^2 (j+3)^{a-n'}.
\end{aligned}$$

Hence, we have proved i)-ix). To see that there exists a convex function  $u$  such that  $f = \nabla u$  we observe that  $Df \in \Gamma_+$  a.e. in  $\Omega$  and we apply the reasoning of [14, Th.1] based on Poincaré's lemma. Therefore, the proof of Theorem 1 is finished.  $\square$

In fact, following the same reasoning as in [23], one can show that our function  $u$  is strictly convex.

Note that the set  $\bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} (A_j^i \cup B_i^j \cup S_{j,i}^a \cup S_{i,j}^a)$  appearing in item iii) at the beginning of the proof of Theorem 1 is the set  $E_{m_1}^j$  mentioned in the Introduction, while  $E_{m_2}^j$  is the inverse of that set.

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